

Advanced Topics in Applied Probability

- Introduction to Lattice Models

Exercises denoted by (★) are harder or use additional theory.

Exercises – Set 6

For subsets $Q_n \subset \mathbb{Z}^d$, we write $Q_n \nearrow \mathbb{Z}^d$ as $n \nearrow \infty$ if the following hold:

- the sequence $(Q_n)_{n \in \mathbb{N}}$ is *increasing*, i.e., $Q_n \subset Q_{n+1}$ for all $n \in \mathbb{N}$, and
- the sequence $(Q_n)_{n \in \mathbb{N}}$ *invades* $\mathbb{Z}^d$, i.e., $\bigcup_{n \in \mathbb{N}} Q_n = \mathbb{Z}^d$.

1. **(Uniqueness of thermodynamic limit)** Consider the weak limit of random-cluster measures on $Q_n := [-n, n]^d \cap \mathbb{Z}^d$, with any fixed boundary conditions ξ (recall Exercise 1(c) in Set 5). Let $\tilde{Q}_n \subset \mathbb{Z}^d$ be another sequence $\tilde{Q}_n \nearrow \mathbb{Z}^d$ as $n \nearrow \infty$. Show that the weak limit is the same:

$$\tilde{\phi}_{p,q}^\xi := \lim_{n \rightarrow \infty} \phi_{p,q;\tilde{Q}_n}^\xi = \lim_{n \rightarrow \infty} \phi_{p,q;Q_n}^\xi =: \phi_{p,q}^\xi.$$

2. **(Equivalence of 2-Potts and Ising models)** Show that the 2-Potts model

$$\pi_{\beta,2}[\sigma] := \frac{1}{Z_{\beta,q}^P} \exp \left(\beta \sum_{\langle u,v \rangle \in E} \delta_{\sigma_u, \sigma_v} \right), \quad \sigma \in \Sigma = \{1, 2\}^V, \quad Z_{\beta,2}^P := \sum_{\sigma \in \Sigma} \exp \left(\beta \sum_{\langle u,v \rangle \in E} \delta_{\sigma_u, \sigma_v} \right),$$

is equivalent to the Ising model

$$\lambda_{\beta/2}[\sigma] := \frac{1}{Z_{\beta/2}^I} \exp \left(\frac{\beta}{2} \sum_{\langle u,v \rangle \in E} \sigma_u \sigma_v \right), \quad \sigma \in \Sigma = \{-1, +1\}^V, \quad Z_{\beta/2}^I := \sum_{\sigma \in \Sigma} \exp \left(\frac{\beta}{2} \sum_{\langle u,v \rangle \in E} \sigma_u \sigma_v \right).$$

3. **(Duality for random-cluster models)** Let $G = (V, E)$ be a finite planar graph and let $G^* = (V^*, E^*)$ be its dual graph. For a random-cluster configuration $\omega \sim \phi_{p,q;G}$, we associate the dual random-cluster configuration ω^* by setting $\omega^*(e^*) := 1 - \omega(e)$ for all $e^* \in E^*$, where $e \in E$ is the unique edge crossed by e^* . Show that

$$\phi_{p,q;G}[\omega] = \phi_{p^*,q^*;G^*}[\omega^*], \quad \text{where} \quad \frac{p^*}{1-p^*} = \frac{q(1-p)}{p} \quad \text{and} \quad q^* = q.$$

[Hint: Euler's formula]

4. **(★) (Critical probability for random-cluster models)** Fix $q \geq 1$. Let

$$p_c(q) = \sup\{p \geq 0 \mid \theta^1(p, q) = 0\} = \sup\{p \geq 0 \mid \theta^0(p, q) = 0\}$$

be the critical value for the random-cluster model, where $\theta^b(p, q) := \phi_{p,q;\mathbb{Z}^2}^b[0 \leftrightarrow \infty]$ for $b \in \{0, 1\}$. (Recall that the equality $p_c(q)$ for both $b \in \{0, 1\}$ follows from convexity of the free energy.)

For example, by using analogous arguments as for percolation ($q = 1$ case), prove that $p_c(q) \geq p_{\text{sd}}(q)$, where $p_{\text{sd}}(q) = \frac{\sqrt{q}}{1+\sqrt{q}}$ is the solution to the *self-duality equation* $p^* = p$.

[Hint: e.g. prove that $\theta^0(p_{\text{sd}}, q) = 0$ using planar duality and the fact that the number of infinite clusters is a.s. 0 or 1.]

5. (**Kramers-Wannier duality for Ising model**) Let $G = (V, E)$ be a finite planar graph, embedded in the plane, and let $\partial^o V := \{v \in V \mid v \text{ belongs to the boundary of the unbounded face of } G\}$. Let g be an additional “ghost” vertex (geometrically, a new vertex added on the unbounded face of G) and set $\partial E := \{\langle v, g \rangle \mid v \in \partial^o V\}$. We define $\bar{G} := (\bar{V}, \bar{E})$ with vertices $\bar{V} = V \cup \{g\}$ and edges $\bar{E} = E \cup \partial E$. Now, fix $\beta \in (0, \infty)$ and consider the Ising model $\lambda_{\beta;G}^\oplus$ on G with plus b.c., with partition function

$$Z_{\beta;G}^\oplus := \sum_{\substack{\sigma \in \{-1, +1\}^{\bar{V}}, \\ \sigma_g = +1 \text{ at } g}} \exp\left(\beta \sum_{\langle u, v \rangle \in \bar{E}} \sigma_u \sigma_v\right).$$

Let $G^* = (V^*, E^*)$ be the dual graph of G , and define $\beta^* \in (0, \infty)$ so that $\tanh \beta^* = e^{-2\beta}$. Show that

$$2^{\#V^*} (\cosh \beta)^{\#E^*} Z_{\beta;G}^\oplus = e^{\beta \#E} Z_{\beta^*;G^*}^{\text{free}},$$

where

$$Z_{\beta^*;G^*}^{\text{free}} := \sum_{\sigma \in \{-1, +1\}^{V^*}} \exp\left(\beta^* \sum_{\langle u, v \rangle \in E^*} \sigma_u \sigma_v\right),$$

is the partition function of the Ising model $\lambda_{\beta^*;G^*}^{\text{free}}$ on G^* with free b.c.

6. (**Average magnetization in the Ising model**) Let $G = (V, E)$ be a finite graph and $\lambda_{\beta,h}$ the Ising model on G (with free b.c.) in an *external magnetic field* of constant magnitude $h > 0$,

$$\lambda_{\beta,h}[\sigma] := \frac{e^{-\beta H_h(\sigma)}}{Z_{\beta,h}}, \quad H_h(\sigma) := - \sum_{\langle u, v \rangle \in E} \sigma_u \sigma_v - h \sum_{v \in V} \sigma_v, \quad \sigma \in \{-1, +1\}^V,$$

with partition function $(\beta, h) \mapsto Z_{\beta,h}$,

$$Z_{\beta,h} := \sum_{\sigma \in \{-1, +1\}^V} e^{-\beta H_h(\sigma)}.$$

- (a) Show that the *average magnetization* $M_{\beta,h}$ in this model (noting that $\lambda_{\beta,h}[\sigma_v]$ is just the expected value of the spin at the vertex v) equals

$$M_{\beta,h} := \frac{1}{\#V} \sum_{v \in V} \lambda_{\beta,h}[\sigma_v] = \frac{1}{\beta \#V} \frac{\partial}{\partial h} \log Z_{\beta,h}.$$

- (b) Show that the *cumulant generating function* of the total magnetization

$$m := \sum_{v \in V} \sigma_v$$

can be expressed as

$$\log \lambda_{\beta,h}[e^{tm}] = \log Z_{\beta,h+t/\beta} - \log Z_{\beta,h},$$

and the r :th cumulant of m is

$$\kappa_r(m) = \frac{1}{\beta^r} \left(\frac{\partial}{\partial h} \right)^r \log Z_{\beta,h}.$$

Upon finding mistakes and/or typos, please contact me!