Advanced Topics in Applied Probability

- Introduction to Lattice Models

Exercises denoted by ($\star$) are harder or use additional theory.

Exercises – Set 5

1. (**Properties of random-cluster model**) Let $p \in (0,1)$ and $q \in (0,\infty)$. Show that, for the random-cluster measure $\phi_{p,q;G}$ on a finite graph $G=(V,E)$, the following hold:

   (a) For each edge $e \in E$ and boundary condition $b \in \{0,1\}$, we have

   $$\phi_{p,q;G}[\omega \,|\, \omega(e)=b] = \begin{cases} \phi_{p,q;G\setminus e}[\omega] & \text{if } b=0, \\ \phi_{p,q;G.e}[\omega] & \text{if } b=1, \end{cases}$$

   where $G \setminus e$ is the graph obtained from G by deleting the edge e and $G.e$ is the graph obtained from G by contracting the edge e.

   (b) (**Positive/finite-energy property**) For $E' \subset E$, write $\{u \overset{E'}{\longleftrightarrow} v\}$ for the event that u and v are connected by an open path in E'. For each edge $e = \langle u,v\rangle \in E$, we have

   $$\phi_{p,q;G}[\omega(e)=1 \,|\, \omega_{E\setminus\{e\}}] = \begin{cases} p & \text{if } \omega_{E\setminus\{e\}} \in \{u \overset{E\setminus\{e\}}{\longleftrightarrow} v\}, \\ \frac{p}{p+q(1-p)} & \text{if } \omega_{E\setminus\{e\}} \notin \{u \overset{E\setminus\{e\}}{\longleftrightarrow} v\}, \end{cases}$$

   where $\omega_{E\setminus\{e\}}$ is the RCM configuration of the edges $E\setminus\{e\}$.
   In particular, we have $\phi_{p,q;G}[\omega(e)=1 \,|\, \omega_{E\setminus\{e\}}] \in (0,1)$, that is, conditional on the values of ω on all edges in $E\setminus\{e\}$, each of the two possible states of e occurs with a strictly positive probability.

   (c) (**Domain Markov property**) For $E' \subset E$, let $G' = (V',E')$ be the induced subgraph with $V' = \{u \in V \,|\, \exists\, v \in V \text{ s.t. } \langle u,v\rangle \in E'\}$. For each $\omega \in \{0,1\}^E$, let $\eta(\omega) = \{e \in E \,|\, \omega(e)=1\}$ and denote by $k(\omega,G')$ the number of clusters of $(V,\eta(\omega))$ that are contained in G'. Suppose X is a random variable defined on E' (i.e., measurable for $\mathcal{F}_{E'} = \sigma\{\omega(e) \,|\, e \in E'\}$). Then,

   $$\phi_{p,q;G}[X \,|\, \mathcal{F}_{E\setminus E'}](\xi) = \phi^{\xi}_{p,q;G'}[X], \qquad \text{for any } \xi \in \{0,1\}^E,$$

   where

   $$\phi^{\xi}_{p,q;G'}[\omega] = \frac{1}{Z^{\xi}_{p,q;G'}} \left(\prod_{e\in E'} p^{\omega(e)}(1-p)^{1-\omega(e)}\right) q^{k(\omega,G')} \qquad \text{if } \omega(e)=\xi(e) \text{ for all } e \in E\setminus E', \quad (1)$$

   and $\phi^{\xi}_{p,q;G'}[\omega]=0$ otherwise, and where

   $$Z^{\xi}_{p,q;G'} = \sum_{\substack{\omega\in\{0,1\}^E: \\ \omega(e)=\xi(e)\ \forall\, e\in E\setminus E'}} \left(\prod_{e\in E'} p^{\omega(e)}(1-p)^{1-\omega(e)}\right) q^{k(\omega,G')}.$$

2. (**FKG inequality**) Let $p \in [0,1]$ and $q \geq 1$. Show that, for the random-cluster measure $\phi_{p,q} = \phi_{p,q;G}$ on a finite graph $G = (V, E)$, the following holds:

$$\phi_{p,q;G}[A \cap B] \geq \phi_{p,q;G}[A]\, \phi_{p,q;G}[B] \qquad \text{if } A \text{ and } B \text{ are increasing events.}$$

[Hint: for example, you may use Holley's inequality (2) from Exercise 5 with $\mu_1 = \phi_{p,q;G}$ and $\mu_2[\cdot] = \phi_{p,q;G}[\cdot \,|\, B]$.]

What goes wrong if $q < 1$?

3. (**Wired percolation probability**) Let $p \in [0,1]$ and $q \geq 1$. Consider the random-cluster measures $\phi^1_{p,q;Q_n}$ with wired boundary conditions on $\Omega = \{0,1\}^{E(\mathbb{Z}^d)}$, each supported on $Q_n := [-n,n]^d \cap \mathbb{Z}^d$ regarded as a graph (V_n, E_n), and defined via (1) by taking $\xi(e) = 1$ for all $e \in E(\mathbb{Z}^d)$ and $E' = E_n$. Let $\phi^1_{p,q} := \lim_{n\to\infty} \phi^1_{p,q;Q_n}$ (weak limit). Show that

$$\theta^1(p,q) := \phi^1_{p,q}[0 \leftrightarrow \infty] = \lim_{n\to\infty} \phi^1_{p,q;Q_n}[0 \leftrightarrow \partial^o V_n],$$

where $\partial^o V_n := \{v \in V_n \,|\, \exists\, u \in \mathbb{Z}^d \setminus V_n \text{ s.t. } \langle u,v\rangle \in E(\mathbb{Z}^d)\}$.

4. (**Comparison inequalities**) Show that, for the random-cluster measure $\phi_{p,q} = \phi_{p,q;G}$ on a finite graph $G = (V, E)$, the following hold:

$$\begin{aligned} \phi_{p',q'} &\leq \phi_{p,q} \qquad \text{if } q' \geq q, \quad q' \geq 1, \quad p' \leq p, \\ \phi_{p',q'} &\geq \phi_{p,q} \qquad \text{if } q' \geq q, \quad q' \geq 1, \quad \frac{p'}{q'(1-p')} \geq \frac{p}{q(1-p)}. \end{aligned}$$

[Hint: for example, you may again use Holley's inequality (2) from Exercise 5.]

5. ($\star$) (**Holley's inequality**) Let $G = (V, E)$ be a finite graph and $\Omega = \{0,1\}^E$.

   (a) Let μ be a positive probability measure on Ω (i.e., $\mu[\omega] > 0$ for all $\omega \in \Omega$). Recall the notations ω^e and ω_e from Exercise 3 of Set 4. Define the generator $Q\colon \Omega \times \Omega \to \mathbb{R}$

   $$Q(\omega_e, \omega^e) := 1, \qquad Q(\omega^e, \omega_e) := \frac{\mu[\omega_e]}{\mu[\omega^e]}, \qquad \text{for all } \omega \in \Omega,\ e \in E,$$

   and $Q(\omega, \omega') = 0$ for other $\omega \neq \omega'$, and finally, choose $Q(\omega, \omega)$ such that

   $$\sum_{\omega' \in \Omega} Q(\omega, \omega') = 0, \qquad \text{for all } \omega \in \Omega.$$

   Show that Q generates an irreducible time-reversible (continuous-time) Markov chain on the state space Ω, whose invariant measure is μ.

   (b) Let μ_1 and μ_2 be positive probability measures on Ω. Assume that

   $$\mu_2[\omega_2^e]\, \mu_1[(\omega_1)_e] \geq \mu_1[\omega_1^e]\, \mu_2[(\omega_2)_e], \qquad \text{for any } e \in E \text{ and } \forall\ \omega_1, \omega_2 \in \Omega \text{ such that } \omega_1 \leq \omega_2. \quad (2)$$

   By defining a Markov chain (X, Y) similar to part (a) on $S = \{(\omega_1, \omega_2) \in \Omega^2 \,|\, \omega_1 \leq \omega_2\}$, show that

   $$\mu_1 \leq \mu_2 \qquad \text{i.e.} \qquad \mu_1[A] \leq \mu_2[A], \qquad \text{for all increasing events } A.$$

[Hint: Note that the stationary measure of X is μ_1 and the stationary measure of Y is μ_2 and that any stationary measure of (X, Y) gives a monotone coupling. Recall Strassen's theorem about monotone couplings and stochastic domination.]

Remark. Usually in the literature, instead of (2) the following *FKG lattice condition* is assumed:

$$\mu_2[\omega_1 \vee \omega_2]\, \mu_1[\omega_1 \wedge \omega_2] \geq \mu_1[\omega_1]\, \mu_2[\omega_2], \qquad \text{for all } \omega_1, \omega_2 \in \Omega,$$

where $(\omega_1 \vee \omega_2)(e) := \max\{\omega_1(e), \omega_2(e)\}$ and $(\omega_1 \wedge \omega_2)(e) := \min\{\omega_1(e), \omega_2(e)\}$.

Upon finding mistakes and/or typos, please contact me!