

Advanced Topics in Applied Probability

- Introduction to Lattice Models

Exercises denoted by $(\star)$ are harder or use additional theory.

Exercises – Set 4

1. (**Self-avoiding walks (SAW)**) A self-avoiding walk (SAW) is a lattice path that visits no vertex more than once. Let $\sigma_n := \#\{\text{SAW of length } n \text{ on } \mathbb{Z}^d \text{ starting from the origin}\}$. Show that $\sigma_{n+m} \leq \sigma_n \sigma_m$ for all $n, m \in \mathbb{N}$, and using this, prove that the following limit exists:

$$d \leq \lim_{n \rightarrow \infty} \sigma_n^{1/n} \leq 2d - 1.$$

Can you find better bounds for this limit?

2. (**Kolmogorov's zero-one law and ergodicity**) Consider a countably infinite graph $G = (V, E)$. Let $\Omega = \{0, 1\}^E$ be endowed with the cylinder sigma-algebra $\mathcal{F}$. Define the tail sigma-algebra as

$$\mathcal{T} := \bigcap_{\substack{E' \subset E \\ \text{finite subset}}} \sigma\{\omega(e) \mid e \notin E'\}.$$

Events $A \in \mathcal{T}$ are called *tail events*.

- (a) Consider percolation on Ω . Show that $\{\exists \text{ an infinite cluster}\}$ is a tail event.
- (b) Show that for any $A \in \mathcal{F}$, there exists a sequence $A_n \in \mathcal{F}$ defined on finite sets $E_n \subset E$ such that $\mathbb{P}[A \Delta A_n] \rightarrow 0$ as $n \rightarrow \infty$, where $A \Delta B = (A \setminus B) \cup (B \setminus A)$ is the symmetric difference.
- (c) Let $(X(e))_{e \in E}$ be i.i.d. random variables on Ω . Using (b), show that $A \in \mathcal{T}$ implies $\mathbb{P}[A] \in \{0, 1\}$.
- (d) Suppose now that $G = \mathbb{Z}^d$. We say that $A \in \mathcal{F}$ is *translation invariant* if

$$A = \pi_x A := \{\pi_x(\omega) \mid \omega \in A\} \quad \text{for all translations } \pi_x(y) := y + x, \quad x, y \in V,$$

where $\pi_x(\omega) := (\omega(\pi_{-x}(e)))_{e \in E}$. A probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ is called *ergodic* if we have $\mathbb{P}[A] \in \{0, 1\}$ for all translation invariant events $A \in \mathcal{F}$. Prove that percolation on Ω is ergodic.

3. (**Influence and Russo's formula**) Consider percolation $\mathbb{P}_p$ on edges of a finite graph $G = (V, E)$ with $p \in (0, 1)$. The *(conditional) influence* of an edge $e \in E$ for an event A is

$$I_A(e) := \mathbb{P}_p[A \mid \omega(e) = 1] - \mathbb{P}_p[A \mid \omega(e) = 0].$$

- (a) Show that if A is an increasing event, then

$$I_A(e) = \mathbb{P}_p[\omega^e \in A] - \mathbb{P}_p[\omega_e \in A] = \mathbb{P}_p[\mathbf{1}_A(\omega^e) \neq \mathbf{1}_A(\omega_e)],$$

$$\text{where} \quad \omega^e(e') = \begin{cases} 1 & \text{if } e' = e, \\ \omega(e') & \text{if } e' \neq e, \end{cases} \quad \omega_e(e') = \begin{cases} 0 & \text{if } e' = e, \\ \omega(e') & \text{if } e' \neq e. \end{cases}$$

- (b) An edge is called *pivotal* for A if $\mathbf{1}_A(\omega^e) \neq \mathbf{1}_A(\omega_e)$. Show that if A is increasing, then this is equivalent to $\omega^e \in A$ and $\omega_e \notin A$. Then show that for any increasing event A , we have

$$\liminf_{\delta \rightarrow 0} \frac{\mathbb{P}_{p+\delta}[A] - \mathbb{P}_p[A]}{\delta} \geq \mathbb{E}[\#\{e \in E \text{ pivotal for } A\}].$$

[Hint: Perturb $\mathbb{P}[\omega(e) = 1]$ first only for edges in a finite box and use Russo's formula there. Then take a limit.]

4. (★) (**Kirchoff's Matrix-Tree Theorem**) Consider Wilson's algorithm on a finite connected graph $G = (V, E)$ for generating a sample of a spanning tree T on G , with some enumeration of the vertices $V = \{v_1, v_2, \dots, v_{n-1}, v_n, t\}$, with t the last (root) vertex, and with transition probabilities $p_{u,v}$ for the random walk. For $A \subset V$, let $\mathcal{G}_A$ be the Green's function (cf. Exercise 5, Set 2).

- (a) Suppose $(\eta_1, \eta_2, \dots, \eta_m)$ are the branches of T produced by the algorithm, generated in this order. Concatenate all of them to obtain the ordered set $\{u_1, u_2, \dots, u_{\ell_m}\}$ of vertices visited by the algorithm; so as vertex-paths, $\eta_1 = (u_1, u_2, \dots, u_{\ell_1})$, $\eta_2 = (u_{\ell_1+1}, u_{\ell_1+2}, \dots, u_{\ell_2})$, $\dots$, so that $u_{\ell_1} = t$ is the root (by Wilson's algorithm) and the last step of each of the other walks $\eta_2, \eta_3, \dots, \eta_m$ belongs to the already generated collection (in particular, the vertices in the ordered set $\{u_1, u_2, \dots, u_{\ell_m}\}$ are not distinct). Show that the probability to obtain this sample is

$$\prod_{k=1}^m \prod_{j=\ell_{k-1}+1}^{\ell_k-1} p_{u_j, u_{j+1}} \mathcal{G}_{A_j}(u_j, u_j),$$

where $A_1 = \{u_0\}$, $A_2 = \{u_0, u_1\}$, $A_3 = \{u_0, u_1, u_2\}$, $\dots$, $A_n = \{u_0, u_1, u_2, \dots, u_{n-1}\}$, writing the root as $u_0 = t$ and using the convention that $\ell_0 = 0$.

- (b) Show that for any $A \subset V$ and $u \in V \setminus A$, we have

$$\mathcal{G}_A(u, u) = \frac{\det(-\Delta^{(A \cup \{u\})})}{\det(-\Delta^{(A)})}$$

where Δ is the Laplacian operator (cf. Exercise 5, Set 2) regarded as an $(|V| \times |V|)$ -matrix, and $\Delta^{(B)}$ is the minor of Δ obtained by removing the rows and columns associated to vertices in B .

- (c) Let $\lambda_1, \lambda_2, \dots, \lambda_{|V|-1}$ be the non-zero eigenvalues of Δ (all are negative). Show that

$$\#\{T \subset G \mid T \text{ is a spanning tree of } G\} = \frac{1}{|V|} \prod_{j=1}^{|V|-1} (-\lambda_j).$$

5. (★) (**Markov Chain - Tree Theorem**) Let X be an irreducible Markov chain in a finite state space S , with associated oriented graph $\vec{H} = (S, \vec{E})$ having vertex set S , edge set $\vec{E} = \{\langle e_-, e_+ \rangle \mid p_{e_-, e_+} > 0\}$, and p_{e_-, e_+} being the transition probabilities of X . The *Markov Chain - Tree Theorem* asserts that the stationary measure of X on S is proportional to

$$\pi_x \propto \sum_{A \in \mathcal{A}_x} \alpha(A) \quad \text{where} \quad \alpha(A) = \prod_{\langle e_-, e_+ \rangle \in A} p_{e_-, e_+}, \quad (1)$$

where $\mathcal{A}_x$ is the set of spanning arborescences of $\vec{H}$ with root x . This can be proven using an auxiliary Markov chain on the space $\mathcal{A} = \prod_{x \in S} \mathcal{A}_x$ as follows. For each $\langle r, x \rangle \in \vec{E}$ and $A \in \mathcal{A}_r$, denote

$$F(A, \langle r, x \rangle) := (A \cup \{\langle r, x \rangle\}) \setminus \{\langle x, y \rangle\} \in \mathcal{A}_x,$$

where y is the unique neighbor vertex of x such that $\langle x, y \rangle \in A$. Define a Markov chain Y on the space $\mathcal{A} = \prod_{x \in S} \mathcal{A}_x$ via the transition probabilities

$$\mathbb{P}[Y_n = F(A, \langle r, x \rangle) \mid Y_{n-1} = A] := p_{r, x}, \quad n \in \mathbb{N}, \quad \langle r, x \rangle \in \vec{E}, \quad A \in \mathcal{A}_r.$$

- (a) Show that the Markov chain Y on $\mathcal{A}$ is irreducible.
(b) Show that the weight $\alpha(A)$ gives the stationary measure for Y .
(c) Conclude that **(1)** gives the stationary measure for the original Markov chain X .

Upon finding mistakes and/or typos, please contact me!