

Advanced Topics in Stochastic Analysis

- Introduction to Schramm-Loewner evolution

Mondays 12–14 and Thursdays 8–10 in *Endenicher Allee 60 - SemR 1.008*

Exercises – Set 11

1. We define the (discrete) *Green's function* for a finite, connected graph $\mathcal{G} = (V, E) \subset \mathbb{Z}^d$ as

$$G_{\mathcal{G}}(x, y) := \mathbb{E}_x \left[\sum_{k=1}^{\tau_{\mathcal{G}}-1} 1\{X_k = y\} \right], \quad x, y \in V,$$

where $(X_n)_{n \in \mathbb{N}}$ is a simple random walk on V started from x and $\tau_{\mathcal{G}} := \inf\{n \in \mathbb{N} \mid X_n \notin V\}$.

- (a) Show that $G_{\mathcal{G}}(x, y) = G_{\mathcal{G}}(y, x)$, for all $x, y \in V$.
 (b) Show that, for fixed $y \in V$, we have $\Delta_x G_{\mathcal{G}}(x, y) = -1\{x = y\}$, where Δ_x denotes the (discrete) Laplacian acting on the variable x , defined as

$$\Delta f(x) := \frac{1}{2d} \sum_{(z, x) \in E} (f(z) - f(x)) \quad f: V \rightarrow \mathbb{R}.$$

2. Let $U \subset \mathbb{C}$ be a bounded domain whose boundary is regular for Brownian motion. Prove that $H_0^1(U) \subset L^2(U)$, where $H_0^1(U)$ is the Hilbert space completion of the space

$$C_0^\infty(U) := \{f: \mathbb{C} \rightarrow \mathbb{R} \mid f \text{ is smooth and } \text{supp}(f) \subset U \text{ is compact}\}$$

with respect to the *Dirichlet inner product*

$$(f, g)_{\nabla} := \frac{1}{2\pi} \int_U \nabla f(z) \nabla g(z) dz.$$

3. Let $\mathcal{G} = (V, E) \subset \mathbb{Z}^d$ be a finite, connected graph. Let h be the (discrete) GFF on $\mathcal{G}$ (with zero b.c.) and let $A \subset V$ be a random subset of vertices, so that we have a coupling (A, h) in the same probability space. We say that the coupling is *local* if the following holds:

For any fixed $B \subset V$, write $h = h_B + h^B$ using the Markov property (for $V \setminus B$), where

- h_B is (discrete) harmonic on $V \setminus B$ and h_B equals h on B ,
- h^B is the GFF on $V \setminus B$ (with zero b.c.), and h^B equals zero on B , and
- h_B and h^B are independent.

Then the process h^B is independent of the sigma-algebra $\sigma(h_B, \{A = B\})$.

Check that in the following situations, the couplings are local:

- (a) For all fixed $B \subset V$, the event $\{A = B\}$ is $\sigma(h_B)$ -measurable.
 (b) Choose $x \in V$ uniformly at random and independently of h , and let A_0 be the largest connected component of V such that $x \in A_0$ and $h(x)h(y) > 0$ for all $y \in A_0$ (cluster of the same sign). Take $A := (A_0 \cup \partial A_0) \cap V$ to be A_0 with one layer added.
4. Let (A_1, h) and (A_2, h) be two local couplings. Assume that conditionally on h , the sets A_1 and A_2 are independent. Show that $(A_1 \cup A_2, h)$ is a local coupling.
5. Consider the (discrete) GFF h on $\mathcal{G} = (V, E) \subset \mathbb{Z}$ with $V = \{-1, 0, 1\}$ and $E = \{(-1, 0), (0, 1)\}$. Let ξ be a random variable independent of h such that $\mathbb{P}[\xi = 1] = \frac{1}{2} = \mathbb{P}[\xi = -1]$. Let $A_1 := \{\xi\}$ and $A_2 := \{\xi \times \text{sign}(h(0))\}$. Show that (A_1, h) and (A_2, h) are local couplings, but $(A_1 \cup A_2, h)$ is not.