

Advanced Topics in Stochastic Analysis

- Introduction to Schramm-Loewner evolution

Mondays 12–14 and **Thursdays 8–10** in *Endenicher Allee 60* - SemR 1.008

Exercises – Set 7

Let B be a standard 1D Brownian motion. Let $\delta \in \mathbb{R}$ and $x > 0$. A *Bessel process* of dimension δ started from x is the solution X to the SDE

$$dZ_t^x = \frac{\delta - 1}{2Z_t^x} dt + dB_t, \quad Z_0^x = x.$$

The solution exists and is unique up to the stopping time (“lifetime”) $T_x := \sup \{t > 0 \mid \inf_{s \in [0, t]} Z_s^x > 0\}$.

1. Let $0 < x < y$ and let $(Z_t^x)_{t \in [0, T_x]}$ and $(Z_t^y)_{t \in [0, T_y]}$ be Bessel processes of dimension $\delta \in \mathbb{R}$ started from $x = Z_0^x$ and $y = Z_0^y$, respectively.

- (a) Let $\lambda > 0$. Show that $t \mapsto \lambda Z_{\lambda^{-2}t}^x$ is a Bessel process of dimension δ started from λx .
- (b) Show that $T_x \leq T_y$ almost surely, and $Z_t^x < Z_t^y$ almost surely for all $t \in [0, T_x)$.
- (c) Show that, almost surely for all $t \in [0, T_x)$, we have

$$\begin{aligned} Z_t^x &\geq B_t + x, & \text{if } \delta \geq 1 \\ Z_t^x &\leq B_t + x, & \text{if } \delta \leq 1. \end{aligned}$$

- (d) Show that $\mathbb{P}\left[\inf_{0 \leq t < T_x} Z_t^x > 0\right] = 1$ if and only if $\delta > 2$.

2. Let $0 < x < y$ and let $(Z_t^x)_{t \in [0, T_x]}$ and $(Z_t^y)_{t \in [0, T_y]}$ be Bessel processes of dimension

$$\delta = 1 + \frac{4}{\kappa} \in (1, \infty), \quad \text{with } \kappa > 0,$$

started from $x = Z_0^x$ and $y = Z_0^y$, respectively. Define

$$R_t := Z_t^y - Z_t^x, \quad U_t := \frac{R_t}{Z_t^y}, \quad N_t := F(U_t),$$

where

$$F(u) := \int_u^1 \frac{ds}{s^{2-8/\kappa}(1-s)^{4/\kappa}}, \quad u \in [0, 1].$$

Calculate the Itô differentials of these processes. In particular, show that N is a local martingale. You may use the fact that

$$F''(u) + \left(\frac{2(1-4/\kappa)}{u} - \frac{4/\kappa}{1-u} \right) F'(u) = 0, \quad u \in (0, 1).$$

3. Let $\kappa \in (4, 8)$ and let $(K_t)_{t \geq 0}$ be the SLE(κ) hulls. Prove that almost surely,

$$\bigcup_{t > 0} \overline{K_t} = \mathbb{H}.$$

4. Let U be a simply connected domain conformally equivalent to $\mathbb{D}$. Fix $z_0 \in U$. Let $\text{crad}_U(z_0) := |f'(0)|$ be the *conformal radius* of U seen from z_0 , where $f: \mathbb{D} \rightarrow U$ is a conformal bijection such that $f(0) = z_0$.

(a) Prove that $\text{crad}_U(z_0)$ is well-defined.

(b) Prove that $\text{crad}_U(z_0)$ is comparable to $\text{dist}(z_0, \partial U)$, that is,

$$\frac{1}{4} \text{crad}_U(z_0) \leq \text{dist}(z_0, \partial U) \leq \text{crad}_U(z_0).$$

[Hint: Koebe]

(c) Let $(g_t)_{t \geq 0}$ be a Loewner chain and $(K_t)_{t \geq 0}$ the corresponding hulls. Show that

$$\text{crad}_{H_t}(z_0) = \frac{2 \text{Im}(g_t(z_0))}{|g_t'(z_0)|}, \quad \text{for } z_0 \in H_t := \mathbb{H} \setminus K_t \text{ and } t \in [0, \tau_{z_0}).$$

5. Let $(g_t)_{t \geq 0}$ be a Loewner chain, $(K_t)_{t \geq 0}$ the corresponding hulls, and $(W_t)_{t \geq 0}$ the driving function.

(a) Prove that

$$\partial_t \log |g_t'(z)| = 2 \frac{(\text{Im}(g_t(z)))^2 - (\text{Re}(g_t(z) - W_t))^2}{\left((\text{Im}(g_t(z)))^2 + (\text{Re}(g_t(z) - W_t))^2 \right)^2}, \quad z \in H_t := \mathbb{H} \setminus K_t \text{ and } t \in [0, \tau_z).$$

(b) Let $z \in \mathbb{H}$. Prove that

$$t \mapsto \log \frac{|g_t'(z)|}{\text{Im}(g_t(z))} := d_t(z)$$

is increasing for $t \in [0, \tau_z)$. Conclude using Problem 4 that

$$\frac{1}{2} \inf_{0 \leq t < \tau_z} \text{dist}(z, \partial H_t) \leq \lim_{t \nearrow \tau_z} e^{-d_t(z)} \leq 2 \inf_{0 \leq t < \tau_z} \text{dist}(z, \partial H_t).$$