Advanced Topics in Stochastic Analysis

- Introduction to Schramm-Loewner evolution

Mondays 12–14 and **Thursdays 8–10** in *Endenicher Allee 60 - SemR 1.008*

Exercises – Set 6

1. Let $h\colon \overline{\mathbb{H}} \to \mathbb{R}$ be a bounded, continuous function, which is harmonic on $\mathbb{H}$. Show that

$$h(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\mathrm{Im}(z)}{|z-u|^2} h(u)\,\mathrm{d}u, \tag{1}$$

and the harmonic conjugate of h is

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\mathrm{Re}(z-u)}{|z-u|^2} h(u)\,\mathrm{d}u + \text{constant}. \tag{2}$$

2. Let $(g_t)_{t\geq 0}$ be a Loewner chain, that is, the solution to

$$\begin{aligned} \partial_t g_t(z) &= \frac{2}{g_t(z) - W_t}, \qquad & z &\in H_t := \{w \in \mathbb{H} \mid t < \tau_w\}, \\ g_0(z) &= z, & z &\in H_0 = \mathbb{H}, \end{aligned} \tag{LE}$$

where $W\colon [0,\infty) \to \mathbb{R}$ is the driving function and $\tau_w = \inf\{s \geq 0 \mid \liminf_{u \nearrow s} |g_u(w) - W_u| = 0\}$.

(a) Show that for all $z \in H_t$, we have

$$\partial_t \mathrm{Im}(g_t(z)) = \frac{-2\,\mathrm{Im}(g_t(z))}{|g_t(z) - W_t|^2}$$

and $t \mapsto \mathrm{Im}(g_t(z))$ is strictly decreasing.

(b) Show that for all $z \in H_t$, we have

$$\partial_t \mathrm{Re}(g_t(z)) = \frac{2\,\mathrm{Re}(g_t(z) - W_t)}{|g_t(z) - W_t|^2}.$$

Fix $T > 0$ and set $M(W) := \sup_{s\in[0,T]}|W_s|$. Show that $t \mapsto \mathrm{Re}(g_t(z))$ is strictly increasing (resp. decreasing) on $[0,T] \ni t$ when $\mathrm{Re}(z) > M(W)$ (resp. $\mathrm{Re}(z) < -M(W)$).

(c) Fix $T > 0$. Let $(h_s)_{s\in[0,T]}$ be the solution to the *backward* LE

$$\partial_s h_s(w) = \frac{-2}{h_s(w) - W_{T-s}}, \qquad h_0(w) = w, \qquad w \in \mathbb{H}.$$

Show that $s \mapsto \mathrm{Im}(h_s(w))$ is strictly increasing. Check that, if $z := h_T(w)$, then $h_{T-t}(w) = g_t(z)$ is the solution to LE with initial condition $g_0(z) = z$.

3. Fix $\kappa > 0$. Let $(g_t)_{t\geq 0}$ be an SLE(κ), i.e., the random Loewner chain with driving process $W_t = \sqrt{\kappa} B_t$, where B is the standard 1D BM. Let $(K_t)_{t\geq 0}$ be the associated hulls and $(\mathcal{F}_t)_{t\geq 0}$ the natural filtration. Show that the following properties hold:

   (a) **Scaling:** For all $\lambda > 0$, we have $(\lambda K_{\lambda^{-2}t})_{t\geq 0} = (K_t)_{t\geq 0}$ in distribution.

   (b) **Conformal Markov property:** For all $s \geq 0$, we have $(\hat{K}_{s,t})_{t\geq 0} = (K_t)_{t\geq 0}$ in distribution and $(\hat{K}_{s,t})_{t\geq 0}$ is independent of $\mathcal{F}_s$, where

   $$\hat{K}_{s,t} := \overline{g_s(K_{s+t} \setminus K_s)} - W_s.$$

   (c) **Strong conformal Markov property:** For any almost surely finite stopping time τ, we have $(\hat{K}_{\tau,t})_{t\geq 0} = (K_t)_{t\geq 0}$ in distribution and $(\hat{K}_{\tau,t})_{t\geq 0}$ is independent of $\mathcal{F}_\tau$.

   (d) **Reflection symmetry:** We have $(m(K_t))_{t\geq 0} = (K_t)_{t\geq 0}$ in distribution, where $m(z) := -\overline{z}$.