Lectures on Probabilistic (correlation function) point of view for (2D $c \leq 1$) CFTs, and SLE random interface models

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Abstract

As is known from the 1980s, CFT should describe scaling limits of critical lattice models in statistical mechanics (i.e., fixed points of the renormalization group flow). One might argue that two-dimensional (2D) CFTs — at least minimal models and Liouville theory — are completely understood since the pioneering works by Belavin–Polyakov–Zamolodchikov [Pol70, Pol81, BPZ84a, BPZ84b], Dorn & Otto [DO94], the Zamolodchikov brothers [ZZ96], Teschner [Tes95], and many others. However, a closer inspection reveals that CFTs relevant to models containing lattice interfaces, or random curves in the continuum, cannot be described by minimal models, seem to exhibit logarithmic phenomena, and even their spectrum (operator content) seems not to be completely clear (see, e.g., [Car13, CR13]).

In the random geometry community, about 25 years ago a breaktrough idea came along: Schramm [Sch00, Sch06] suggested that probabilistically, such interface models could be rigorously described by combining classical complex analysis (namely, Loewner theory [Loe23]) with stochastic analysis (namely, Brownian motion and martingale theory). This provided a wonderful description of scaling limits of lattice interfaces, and turned out to have deeper roots than perhaps originally anticipated. Indeed, Schramm's random SLE curves also share an intrinsic connection with the geometric and algebraic content in CFT. On the one hand, such curves emanating at boundary or bulk points relate to specific Virasoro modules in the theory — in particular, in the case of boundary phenomena they correspond to degenerate field insertions, which can be studied completely rigorously in terms of probability and PDE theory. (I plan to focus on this connection, which is simpler and more developed.) On the other hand, the action functional for SLE loops is closely related to the universal Liouville action and Kähler geometry on Teichmüller space, although the associated correlation functions therein remain more mysterious. (Hence, I plan not to discuss this connection.)

I will try to describe how correlation functions in 2D CFTs (with central charge $c \leq 1$) can be concretely understood in terms of the random interface models, giving rise to a complete description of crossing probabilities and chiral boundary/diagonal-bulk conformal blocks for degenerate fields. I explain some emergent quantities in the semiclassical limit $c \to -\infty$, which appear as accessory parameters for certain geometric problems in Teichmüller theory, and whose dynamics is described by classical integrable Calogero-Moser type systems.

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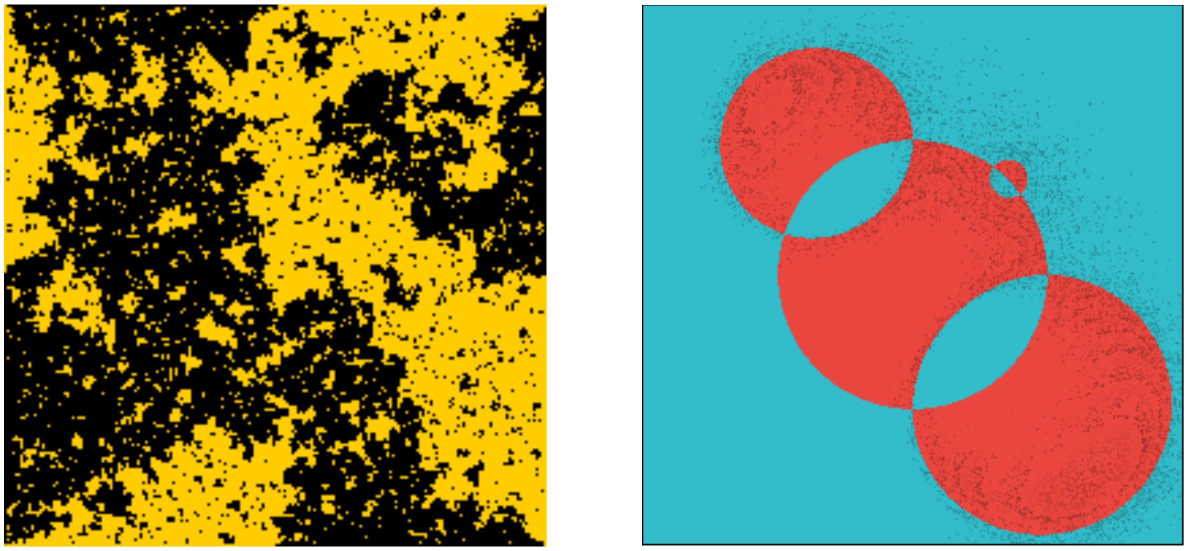
These **rough** lecture notes describe parts of contents of the minicourse "Probabilistic point of view for CFT" at Pascal Institute (Orsay, fall 2025). **Please let me know if you find mistakes or misprints!** Also, many references are missing; please let me know if you have suggested additions!



Left: A configuration of the critical Ising model. Right: A configuration of a semiclassical limit of SLE curves.

Contents

1 Introduction — What is CFT?

These lectures are targeted to a mixed audience working on different aspects of CFT. To highlight the ideas and remove technicalities, the text is presented in a narrative manner, and some citations for precise results are given along the way (more might be added later).

The presentation is not meant to be a complete account on the topic — I wonder if one can ever say that CFT has been completely understood and an complete account could be provided...

1.1 What is CFT?

There are many definitions for "conformal field theory" (CFT), with many differences and similarities (and none are probably completely equivalent). For example, a "CFT" might refer to:

- ▷ Quantum field theory (QFT) with additional conformal symmetry.
- ▷ Vertex operator algebra (VOA) and a module for it.
- ▷ Representation theory of Virasoro algebra (or extended symmetry algebras containing it).
- ▷ (Segal) functor from Riemann surfaces (cobordisms) to vector spaces (or modules, etc.).
- ▷ (Feynman) path integral that is conformally invariant.
- ▷ Scaling limit of critical lattice model from statistical mechanics.
- ▷ Collection of consistent correlation functions (possibly with a Virasoro action).
- ▷ Something else?

Everyone working on this area can decide which point of view is their favorite. Or (like me) resort to working with whichever point of view is appropriate for each situation. In these lectures, we will focus on aspects of the correlation function point of view for 2D CFTs with central charge $c \leq 1$. These are motivated on the one hand from scaling limits of critical lattice models and on the other hand from models for random conformally invariant curves (SLEs).

1.2 Correlation functions in CFT

This section is meant for refreshing background material for the purposes of the correlation function point of view that we are going to take in these lectures. **For readers familiar with CFT, this section can be easily skipped — and one can come back to it to when needed.** (One can safely proceed to Section 2 where the main content of the lectures begins.)

Let us emphasize that in CFT, the *fields* themselves might not be analytically well-defined objects, but nevertheless, their *correlation functions* are well-defined functions. As a concrete example, Liouville CFT was recently constructed completely rigorously [DKRV16, KRV20] combining the ideas of [Tes95] (and others) with a very powerful probabilistic technique. However, for instance for some objects in the critical Ising model (discussed below), the CFT description is not completely clear [GK25]. (A scaling limit of the Ising energy field should morally be a product of two spin fields — but how does one multiply random distributions?)

Thus, we will be mainly interested in correlation functions in CFT, which describe — in some sense — the physical observables in the models of interest. Let us focus on chiral theory on the Riemann sphere $\hat{\mathbb{C}}$ or a subset of it. Then, correlation functions are analytic (multi-valued) functions $F : \mathfrak{W}_n \to \mathbb{C}$ (also called n-point functions) defined on the configuration space

$$\mathfrak{W}_n := \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j \text{ if } i \neq j\}.$$

Physicists speak of correlation functions as "vacuum expectation values" of (primary) fields $\phi_{\iota_i}(z_i)$ labeled by some indices ι_i and denote them by

$$F_{\iota_1,\ldots,\iota_n}(z_1,\ldots,z_n) = \langle \phi_{\iota_1}(z_1)\cdots\phi_{\iota_n}(z_n)\rangle.$$

Because of the conformal symmetry, the correlation functions are assumed to be covariant under (global) conformal transformations. In a CFT on the full Riemann sphere $\hat{\mathbb{C}} = \mathbb{C}\cup\{\infty\}$, this means that under all Möbius transformations[1] $f \in \mathrm{PSL}(2,\mathbb{C})$, we have

$$F_{\iota_1,\ldots,\iota_n}(z_1,\ldots,z_n) = \prod_{i=1}^{n} |f'(z_i)|^{\Delta_{\iota_i}} \times F_{\iota_1,\ldots,\iota_n}(f(z_1),\ldots,f(z_n)),$$

with some *conformal weights* $\Delta_{\iota_i} \in \mathbb{R}$ associated to the fields ϕ_{ι_i}.

Notably, global conformal invariance only results in *finitely* many (three) constraints for the physical system. However, Belavin, Polyakov, and Zamolodchikov (BPZ) observed in the 1980s that, in two dimensions, imposing *local* conformal invariance yields *infinitely many independent symmetries* [BPZ84a, BPZ84b]. On $\hat{\mathbb{C}}$, the local conformal transformations are just the locally invertible holomorphic and anti-holomorphic maps — see, e.g., [Sch08, Chapters 1,2,5] for details.

1.3 Conformal symmetry and Virasoro algebra

Roughly speaking, in CFT à la BPZ, one regards the local conformal invariance as invariance under infinitesimal transformations (or vector fields which generate the local conformal mappings): for instance, the infinitesimal holomorphic transformations are written as Laurent series,

$$z \mapsto z + \sum_{n\in\mathbb{Z}} a_n z^n,$$

which can be seen to be generated by the vector fields

$$\ell_n := -z^{n+1}\frac{\partial}{\partial z}, \qquad n \in \mathbb{Z},$$

constituting a Lie algebra isomorphic to the *Witt algebra* $\mathfrak{Witt}$ with commutation relations

$$[\ell_n, \ell_m] = (n-m)\ell_{n+m}.$$

Note that ℓ_{-1} is the infinitesimal generator of translations, ℓ_0 of scalings, and ℓ_1 of special conformal transformations. The Lie subalgebra generated by $\{\ell_{-1}, \ell_0, \ell_1\}$ is isomorphism to the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ of the Möbius group $\mathrm{PSL}(2,\mathbb{C})$ of global conformal transformations on $\hat{\mathbb{C}}$.

In quantized systems, the symmetry groups and algebras often are *central extensions* of their classical counterparts. In particular, in conformally invariant quantum field theory (i.e., CFT), the conformal symmetry algebra is the unique central extension of the Witt algebra by the one-dimensional abelian Lie algebra $\mathbb{C}$: namely the *Virasoro algebra* $\mathfrak{Vir}$. (The central part of $\mathfrak{Vir}$ represents a *conformal anomaly*, giving rise to a projective representation of $\mathfrak{Witt}$ — see, e.g. [Sch08, Sections 3-4] for details.) $\mathfrak{Vir}$ is the infinite-dimensional Lie algebra generated[2] by L_n, for $n \in \mathbb{Z}$, together with a central element C, with commutation relations

$$\begin{cases} [\mathrm{L}_n, \mathrm{L}_m] = (n-m)\mathrm{L}_{n+m} + \frac{1}{12}n(n^2-1)\delta_{n,-m}\mathrm{C}, & \text{for } n,m \in \mathbb{Z}, \\ [\mathrm{L}_n, \mathrm{C}] = 0. \end{cases}$$

Algebraically, the basic objects in a CFT, the conformal fields, can be regarded as elements in *representations* of the symmetry algebra $\mathfrak{Vir}$, where the central element acts as a constant multiple of the identity, $\mathrm{C} = c\,\mathrm{id}$. The number $c \in \mathbb{C}$ is called the *central charge* of the CFT.

[1] Of specific interest to us will be CFT in the domain $\mathbb{H}$ with boundary $\partial\mathbb{H} = \mathbb{R}$, where the global conformal transformations are also Möbius maps, $f \in \mathrm{PSL}(2,\mathbb{R})$.

[2] We will use the same notation $\mathfrak{Vir}$ also for the universal enveloping algebra of the Virasoro algebra.

1.4 Primary fields and Verma modules

Primary fields are fields whose correlation functions also have a covariance property also under *local* conformal transformations, in an "infinitesimal" sense, see [Sch08, Chapter 9]. Other fields[3] are called descendant fields, obtained from the primary fields by action of the Virasoro algebra.

The rough idea is the following. It is postulated that a primary field $\phi_\iota(z)$ of conformal weight Δ_ι generates a *highest-weight module* $\mathsf{V}_{c,\Delta_\iota}$ of the Virasoro algebra of *weight* Δ_ι and *central charge* c. In physics, it is called the *conformal family* of $\phi_\iota(z)$, consisting of linear combinations of the *descendant fields* of $\phi_\iota(z)$. The latter are obtained from $\phi_\iota(z)$ via action of the Virasoro algebra. Here the space-time point $z \in \hat{\mathbb{C}}$ plays no role yet. Algebraically, we could identify $\phi_\iota(z)$ with a highest-weight vector as defined below:

$$\phi_\iota(z) \text{ with weight } \Delta_\iota \qquad \Longleftrightarrow \qquad v_{c,h} \text{ with weight } h = \Delta_\iota .$$

We use the algebraic notation from the right-hand side when discussing representations of $\mathfrak{Vir}$, and the analytical notation from the left-hand side when discussing fields in a CFT.

A $\mathfrak{Vir}$-module V is a *highest-weight module* if

$$V = \mathfrak{Vir}\, v_{c,h},$$

where $v_{c,h} \in V$ is a *highest-weight vector* of weight $h \in \mathbb{C}$ and central charge $c \in \mathbb{C}$, that is, a vector satisfying

$$\mathrm{L}_0 v_{c,h} = h v_{c,h}, \qquad \mathrm{L}_n v_{c,h} = 0, \qquad \text{for } n \geq 1, \qquad \text{and} \qquad \mathrm{C} v_{c,h} = c v_{c,h}.$$

In particular, for any pair (c,h), there exists a unique (up to isomorphism) *Verma module*

$$\mathrm{M}_{c,h} = \mathfrak{Vir}/\mathrm{I}_{c,h}$$

where $\mathrm{I}_{c,h}$ is the left ideal generated by the elements $\mathrm{L}_0 - h1$, $\mathrm{C} - c1$, and L_n, for $n \geq 1$. The Verma module $\mathrm{M}_{c,h}$ is a highest-weight module generated by a highest-weight vector $v_{c,h}$ of weight h and central charge c (given by the equivalence class of the unit 1). It has a Poincaré-Birkhoff-Witt type basis given by the action of the Virasoro generators with negative index,

$$\left\{ \mathrm{L}_{-n_1} \cdots \mathrm{L}_{-n_k} v_{c,h} \mid n_1 \geq \cdots \geq n_k > 0,\, k \in \mathbb{Z}_{\geq 0} \right\}$$

ordered by applying the commutation relations. The Verma modules $\mathrm{M}_{c,h}$ are *universal* in the sense that if V is any $\mathfrak{Vir}$-module containing a highest-weight vector v of weight h and central charge c, then there exists a canonical homomorphism $\varphi : \mathrm{M}_{c,h} \to V$ such that $\varphi(v_{c,h}) = v$. In other words, any highest-weight $\mathfrak{Vir}$-module is isomorphic to a quotient of some Verma module.

1.5 Descendant fields and BPZ PDEs

Suppose that the primary field $\phi_\iota(z)$ is given. In general, its descendants have the form

$$\Psi(z) \; = \; \mathrm{L}_{-n_1} \cdots \mathrm{L}_{-n_k} \phi_\iota(z), \qquad \text{where} \quad n_1 \geq \cdots \geq n_k > 0 \text{ and } k \geq 1.$$

Their correlation functions are formally determined from the correlation functions of $\phi_\iota(z)$ using linear differential operators which arise from the generators of the Virasoro algebra (this is quite complicated — see, e.g., [Mus10, Chapter 10]): for any primary fields $\{\phi_{\iota_i}(z_i) \mid 1 \leq i \leq n\}$,

$$\left\langle \phi_{\iota_1}(z_1) \cdots \phi_{\iota_n}(z_n)\, \mathrm{L}_{-k} \phi_\iota(z) \right\rangle \overset{(\star)}{=} \mathcal{L}_{-k}^{(z)} \left\langle \phi_{\iota_1}(z_1) \cdots \phi_{\iota_n}(z_n) \phi_\iota(z) \right\rangle,$$

[3] There is also the special field called stress-energy tensor, that we won't discuss here.

where

$$\mathcal{L}_{-k}^{(z)} := \sum_{i=1}^{n} \left(\frac{(k-1)\Delta_{\iota_i}}{(z_i - z)^k} - \frac{1}{(z_i - z)^{k-1}} \frac{\partial}{\partial z_i} \right), \qquad \text{for } k \in \mathbb{Z}_{>0}. \tag{1.1}$$

Here, the identity ($\star$) should be thought of as a "black box", that is heuristically argued in the physics literature [Mus10, Chapter 10] via the "infinitesimal conformal symmetry" of the space-time, and can be a posteriori rigorously verified in some cases, such as for the Liouville theory [KRV19]. One can extend the property ($\star$) also to any (descendant) field $\Psi(z)$:

$$\langle \phi_{\iota_1}(z_1) \cdots \phi_{\iota_n}(z_n)\, \mathrm{L}_{-k}\Psi(z) \rangle \quad \overset{(\star)}{=} \quad \mathcal{L}_{-k}^{(z)} \langle \phi_{\iota_1}(z_1) \cdots \phi_{\iota_n}(z_n)\Psi(z) \rangle.$$

Upshot. The conclusion from here is that the linear differential operators (1.1) relate the *purely algebraic content* in CFT, encoded in representations of the Virasoro algebra $\mathfrak{Vir}$, to its *analytical content* that includes the dependence of the space-time variables $z_1, \ldots, z_n, z \in \mathfrak{W}_{n+1}$.

Now, let's consider the $\mathfrak{Vir}$-module $\mathsf{V}_{c,\Delta_\iota}$ generated by the primary field $\phi_\iota(z)$ with weight Δ_ι. We know that it is must be quotient of a Verma module by some submodule J_ι:

$$\mathsf{V}_{c,\Delta_\iota} \cong \mathrm{M}_{c,\Delta_\iota}/\mathrm{J}_\iota.$$

Of course, the quotient structure needs to be determined from some information about $\phi_\iota(z)$. We could have $\mathrm{J}_\iota = \{0\}$ or $\mathrm{J}_\iota = \mathrm{M}_{c,\Delta_\iota}$, in which case there's nothing to quotient by. However, in certain special cases we have a non-trivial quotient, which results in interesting information about correlations of $\phi_\iota(z)$ with other fields. (See Section 1.6 for classification of those cases.)

Suppose that the conformal weight $\Delta_\iota = h_{r,s}$ belongs to the special class (1.5) discussed below, and denote $\phi_\iota := \phi_{r,s}$ accordingly. Then, by Theorem 1.1 (stated in the next Section 1.6), the Verma module $\mathrm{M}_{c,h_{r,s}}$ contains a so-called *singular* vector (defined below)

$$v = P(\mathrm{L}_{-1}, \mathrm{L}_{-2}, \ldots) v_{c,h_{r,s}} \quad \in \quad \mathrm{M}_{c,h_{r,s}}$$

at level rs, where P is a polynomial in the generators of the Virasoro algebra.

Suppose furthermore that[4] this vector is contained in $\mathrm{J}_{r,s}$:

$$v = P(\mathrm{L}_{-1}, \mathrm{L}_{-2}, \ldots) v_{c,h_{r,s}} \quad \in \quad \mathrm{J}_{r,s}. \tag{1.2}$$

Then, its equivalence class in the quotient module is zero:

$$[v] = 0 \quad \in \quad \mathrm{M}_{c,h_{r,s}}/\mathrm{J}_{r,s} \cong \mathsf{V}_{c,h_{r,s}}. \tag{1.3}$$

In other words, the descendant field

$$P(\mathrm{L}_{-1}, \mathrm{L}_{-2}, \ldots)\phi_{r,s}(z) = 0$$

corresponding to the singular vector v is zero, a "null-field". In this case, we say that $\phi_{r,s}(z)$ has a *degeneracy at level rs*. In particular, correlation functions containing the field $\phi_{r,s}(z)$ then satisfy partial differential equations (known as "null-field equations") given by the polynomial

$$P(\mathcal{L}_{-1}^{(z)}, \mathcal{L}_{-2}^{(z)}, \ldots)$$

and the differential operators (1.1):

$$\begin{aligned} 0 \;&=\; \langle \phi_{\iota_1}(z_1) \cdots \phi_{\iota_n}(z_n) P(\mathrm{L}_{-1}, \mathrm{L}_{-2}, \ldots)\phi_{r,s}(z) \rangle && \text{[by (1.3, 1.2)]} \\ &\overset{(\star)}{=}\; P(\mathcal{L}_{-1}^{(z)}, \mathcal{L}_{-2}^{(z)}, \ldots) \langle \phi_{\iota_1}(z_1) \cdots \phi_{\iota_n}(z_n)\phi_{r,s}(z) \rangle && \text{[by "black box" } (\star)] \end{aligned}$$

Upshot. For the correlation function with $\phi_\iota(z) = \phi_{r,s}(z)$, we see that from certain *linear relations* on the Virasoro module side, we obtain the following (perfectly well-defined) *partial differential equation* (called BPZ PDE) on the correlation function side:

$$F_{\iota_1,\ldots,\iota_n,\iota} : \mathfrak{W}_{n+1} \to \mathbb{C}, \qquad P(\mathcal{L}_{-1}^{(z)}, \mathcal{L}_{-2}^{(z)}, \ldots)\, F_{\iota_1,\ldots,\iota_n,\iota}(z_1, \ldots, z_n, z) = 0. \tag{1.4}$$

[4] This is the case, e.g., when $\mathsf{V}_{c,h_{r,s}}$ is irreducible.

1.6 Structure of Verma modules and singular vectors for $\mathfrak{Vir}$

Let us summarize here some facts concerning the representation theory of the Virasoro algebra $\mathfrak{Vir}$. In general, submodules of Verma modules were classified by Feĭgin & Fuchs [FF84, IK11].

▷ A vector $v \in \mathrm{M}_{c,h} \smallsetminus \{0\}$ is said to be *singular* at level $\ell \in \mathbb{Z}_{>0}$ if it satisfies

$$\mathrm{L}_0 v = (h + \ell) v \qquad \text{and} \qquad \mathrm{L}_n v = 0, \qquad \text{for } n \geq 1.$$

▷ Each Verma module $\mathrm{M}_{c,h}$ has a unique maximal proper submodule, and the quotient of $\mathrm{M}_{c,h}$ by this submodule is the unique irreducible highest-weight $\mathfrak{Vir}$-module of weight h and central charge c. (This is analogous to the classical theory of Lie algebras.)

▷ Every non-trivial submodule of a Verma module $\mathrm{M}_{c,h}$ is generated by some singular vectors.

▷ The L_0-eigenvalue of a basis vector $v = \mathrm{L}_{-n_1} \cdots \mathrm{L}_{-n_k} v_{c,h} \in \mathrm{M}_{c,h}$ can be calculated using the commutation relations:

$$\mathrm{L}_0 v = (h + \sum_{i=1}^{k} n_i) v = (h + \ell) v.$$

The number $\ell := \sum_{i=1}^{k} n_i$ is called the *level* of the vector v.

In particular, Feĭgin and Fuchs found a characterization for the existence of singular vectors and thus for the irreducibility of $\mathrm{M}_{c,h}$. Indeed, the Verma module $\mathrm{M}_{c,h}$ is irreducible if and only if it contains no singular vectors. On the other hand, $\mathrm{M}_{c,h}$ contains singular vectors precisely when the numbers (c, h) belong to a special class:

Theorem 1.1. [FF84, Proposition 1.1 & Theorem 1.2] *The following are equivalent:*

1. *The Verma module* $\mathrm{M}_{c,h}$ *contains a singular vector.*
2. *There exist* $r, s \in \mathbb{Z}_{>0}$*, and* $\theta \in \mathbb{C} \smallsetminus \{0\}$ *such that*

$$\begin{cases} h = h_{r,s}(\theta) := \frac{(r^2-1)}{4}\theta + \frac{(s^2-1)}{4}\theta^{-1} + \frac{(1-rs)}{2}, \\ c = c(\theta) = 13 - 6(\theta + \theta^{-1}). \end{cases} \tag{1.5}$$

In this case, the smallest such $\ell = rs$ *is the lowest level at which a singular vector occurs in* $\mathrm{M}_{c,h}$*.*

The special conformal weights $h_{r,s}$ are the roots of the Kac determinant [Kac79, Kac80], often called *Kac conformal weights.* The notation $h_{r,s}$ for them is very common historically.

▷ $\mathrm{L}_{-1} v_{c,h}$ is a singular vector at level one if and only if $h = h_{1,1} = 0$.

▷ As a more involved example, let us make an ansatz

$$v = (\mathrm{L}_{-2} + a\mathrm{L}_{-1}^2) v_{c,h} \tag{1.6}$$

for a singular vector at level two, with some $a \in \mathbb{C}$. Then, we must have

$$a = -\frac{3}{2(2h+1)}, \qquad h = \frac{1}{16}\big(5 - c \pm \sqrt{(c-1)(c-25)}\big),$$

which equals $h_{1,2}$ or $h_{2,1}$ depending on the choice of sign.

In general, explicit expressions for singular vectors are hard to find — one has to construct a suitable (complicated) polynomial P so that the vector $v = P(\mathrm{L}_{-1}, \mathrm{L}_{-2}, \ldots)v_{c,h}$ is singular. Remarkably, in the case when either $r = 1$ or $s = 1$, Benôit and Saint-Aubin found a family of such vectors [BSA88]: for $r = 1$ and $s \in \mathbb{Z}_{>0}$, the singular vector at level $\ell = s$ has the formula

$$\sum_{k=1}^{s} \sum_{\substack{n_1,\ldots,n_k \geq 1 \\ n_1+\cdots+n_k=s}} \frac{(-\theta)^{k-s}\,(s-1)!^2}{\prod_{j=1}^{k-1}(\sum_{i=1}^{j} n_i)(\sum_{i=j+1}^{k} n_i)} \times \mathrm{L}_{-n_1}\cdots\mathrm{L}_{-n_k} v_{c,h_{1,s}}.$$

The case $s = 1$ and $r \in \mathbb{Z}_{>0}$ is obtained by taking $\theta \mapsto \theta^{-1}$. Later, Bauer, Di Francesco, Itzykson, and Zuber found the general singular vectors via a fusion procedure [BFIZ91].

1.7 Examples of BPZ PDEs

Singular vectors give rise to kind of degeneracies in CFT — null-fields whose correlation functions solve BPZ PDEs (1.4) obtained from the Virasoro generators.

▷ From the singular vector at level one, one obtains the null-field $\mathrm{L}_{-1}\phi_{1,1}(z)$, whose correlation functions $F_{\iota_1,\ldots,\iota_n,\iota}(z_1,\ldots,z_n,z) := \langle \phi_{\iota_1}(z_1)\cdots\phi_{\iota_n}(z_n)\phi_{1,1}(z)\rangle$ satisfy the PDE

$$0 \quad = \quad \mathcal{L}_{-1}^{(z)}\, F_{\iota_1,\ldots,\iota_n,\iota}(z_1,\ldots,z_n,z) \quad = \quad -\sum_{i=1}^{n} \frac{\partial}{\partial z_i} F_{\iota_1,\ldots,\iota_n,\iota}(z_1,\ldots,z_n,z).$$

Assuming that the correlation function F is translation invariant, we can replace $\sum_{i=1}^{n} \frac{\partial}{\partial z_i}$ by the single derivative $-\frac{\partial}{\partial z}$, so

$$\frac{\partial}{\partial z} F_{\iota_1,\ldots,\iota_n,\iota}(z_1,\ldots,z_n,z) = 0,$$

i.e., the correlation function is *constant* in the variable z corresponding to $\phi_{1,1}(z)$.

▷ More interestingly, for the level two singular vectors (1.6), the corresponding null-fields are

$$\Big(\mathrm{L}_{-2} - \frac{3}{2(2h_{1,2}+1)}\mathrm{L}_{-1}^2\Big)\phi_{1,2}(z),$$
$$\Big(\mathrm{L}_{-2} - \frac{3}{2(2h_{2,1}+1)}\mathrm{L}_{-1}^2\Big)\phi_{2,1}(z).$$

In the former case, the correlation functions $F_{\iota_1,\ldots,\iota_n,\iota}(z_1,\ldots,z_n,z) := \langle \phi_{\iota_1}(z_1)\cdots\phi_{\iota_n}(z_n)\phi_{1,2}(z)\rangle$ satisfy the second order PDE

$$\left[-\frac{3}{2(2h_{1,2}+1)}\left(\sum_{i=1}^{n}\frac{\partial}{\partial z_i}\right)^2 - \sum_{i=1}^{n}\left(\frac{1}{z_i - z}\frac{\partial}{\partial z_i} - \frac{\Delta_{\iota_i}}{(z_i - z)^2}\right)\right] F_{\iota_1,\ldots,\iota_n,\iota}(z_1,\ldots,z_n,z) = 0,$$

where Δ_{ι_i} are the conformal weights of the fields ϕ_{ι_i}, for $1 \leq i \leq n$. Assuming again translation invariance, this PDE simplifies to

$$\left[\frac{3}{2(2h_{1,2}+1)}\frac{\partial^2}{\partial z^2} - \sum_{i=1}^{n}\left(\frac{1}{z_i - z}\frac{\partial}{\partial z_i} - \frac{\Delta_{\iota_i}}{(z_i - z)^2}\right)\right] F_{\iota_1,\ldots,\iota_n,\iota}(z_1,\ldots,z_n,z) = 0. \tag{1.7}$$

Using the parameterization $\theta = \kappa/4$, we have $c = \frac{(3\kappa-8)(6-\kappa)}{2\kappa}$ and $h_{1,2} = \frac{6-\kappa}{2\kappa}$. Then, the PDE (1.7) is the same as we will see in SLE(κ) theory.

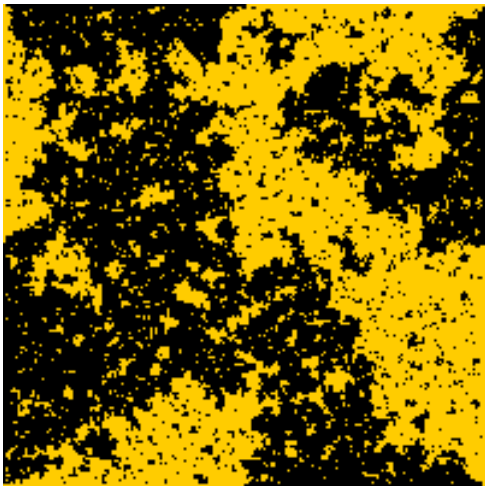


Figure 2.1: A configuration of the critical Ising model on a square with 8 alternating boundary segments with fixed spin "+" (yellow) or "−" (black). One can see 4 macroscopic interfaces between the fixed boundary points where the boundary conditions change. (Figure from [PW19].)

2 Random geometric observables in CFT

The content of the Pascal Institute lectures begins here.

2.1 Interfaces in critical lattice models (Ising example)

For concreteness let us consider the spin Ising model, which describes a magnet with a paramagnetic (disordered) and a ferromagnetic (ordered) phase. See, e.g., the lecture notes [DCS12]. While the model could be defined on any graph, we are interested in 2D systems. For simplicity, let us consider the model on subsets of the square lattice $\mathbb{Z}^2$ (re-scaled by small $\delta > 0$).

Fix a bounded simply connected domain $D \subsetneq \mathbb{C}$ of the complex plane. Let $G = (V, E)$ be the a finite graph with vertices $V = D \cap \delta\mathbb{Z}^2$ and edges E given by the nearest-neighbor pairs. A configuration in the (ferromagnetic) Ising model consists of an assignment $\sigma : V \to \{\pm 1\}$ of spins $\sigma_x \in \{\pm 1\}$ to each vertex $x \in V$. The probability of a configuration σ is given by the *Boltzmann distribution* (the canonical ensemble, or Gibbs measure)

$$\mathbb{P}[\sigma] = \frac{e^{-\beta H(\sigma)}}{Z}, \qquad H(\sigma) = -\sum_{(x,y)\in E} \sigma_x \sigma_y, \qquad Z = \sum_{\sigma} e^{-\beta H(\sigma)},$$

where $\beta = \frac{1}{T} > 0$ is the inverse-temperature and $H(\sigma)$ is the Hamiltonian (giving the interaction).

The behavior of the system is highly dependent on the temperature: there is an order-disorder phase transition at a unique[5] critical temperature $\beta_c = \frac{1}{T_c} \in (0, \infty)$. At the critical temperature, the scaling limit of the Ising model is believed (and in many ways proved) to become conformally invariant in the scaling limit [Pol70, BPZ84b] (e.g., its interfaces and correlation functions converge to conformally invariant or covariant quantities [HS13, CHI15, CDCH+14, Izy17, BPW21, CHI21]). We will only consider the Ising model at its critical temperature.

To study the geometry of the Ising model, one can study *interfaces* between "+" spins and "−" spins. Some of these interfaces are macroscopic, so they survive in the scaling limit. For example, one can force the system to have a macroscopic interface via imposing boundary conditions. We split the boundary $\partial D = \partial^+ \sqcup \partial^-$ into two segments ∂^+ and ∂^-, and consider the Ising model with the constraint that the vertices in ∂^+ all equal $+1$ and the vertices in ∂^- all equal -1.

[5] The precise value of the critical temperature depends on the chosen graph. For $\mathbb{Z}^2$ it is $\beta_c = \frac{1}{2}\log(1+\sqrt{2})$.

Then for topological reasons, there must exist[6] a macroscopic path traversing between "+" and "−" spins and connecting the two boundary points where the segments ∂^+ and ∂^- touch. Of course, one could generalize this to include several alternating "+" and "−" boundary segments, and obtain several (interacting) interfaces. In the scaling limit $\delta \to 0$, such interfaces have been proven to converge to random conformally invariant curves [CDCH+14, Izy17, BPW21], called Schramm-Loewner evolution (SLE(3)) curves, that we will discuss shortly.

2.2 Ising CFT — $(4,3)$ minimal model

Let us briefly discuss correlations of the spins σ_x at vertices $x \in V$ and the energy operators $\varepsilon_{(x,y)} := \sigma_x \sigma_y$ at edges $(x,y) \in E$. It is a major achievement in this area that (multi-point) correlation functions of these random variables converge to conformally covariant quantities that satisfy all the properties predicted for the Ising model CFT [HS13, CHI15, CHI21]. Such random variables should in the scaling limit correspond to CFT fields, whose nature however remains partly unclear: while the *spin field* σ can be realized as a random distribution [CGN15], it seems that the *energy field* ε cannot be realized in that way [GK25]. Since we will not focus on this, we refer to the above literature for more details on the mathematical results.

Let us however make a connection with the theoretical physics literature (e.g. [DFMS97]). As was already understood in the 1980s, the observables $\{\sigma, \varepsilon, 1\}$ in the Ising model correspond with the three CFT primary fields in the $(4,3)$-minimal model. This model carries the Virasoro symmetry, and its states can be regarded as elements in the $\mathfrak{Vir} \times \overline{\mathfrak{Vir}}$-module

$$(M_{1/16} \otimes \overline{M}_{1/16}) \oplus (M_{1/2} \otimes \overline{M}_{1/2}) \oplus (M_0 \otimes \overline{M}_0),$$

where[7] each M_h (resp. $\overline{M}_{\bar{h}}$) is a (simple) highest weight $\mathfrak{Vir}$-module of weight h (resp. $\overline{\mathfrak{Vir}}$-module of weight $\bar{h}$) and central charge $c = 1/2$. In particular, the Ising minimal model CFT is diagonal and unitary. For the $\mathfrak{Vir}$-structure related to the Ising model at the lattice level, see [HJVK22]. For its scaling limit, see [CHI21] (to my understanding, the authors are working on a preprint verifying the Virasoro structure as well).

As a simple example, the two-point function (in the full plane $D = \mathbb{C}$) reads

$$\begin{aligned} \delta^{-1/4}\Big(\mathbb{E}[\sigma_x \sigma_y] - \mathbb{E}[\sigma_x]\mathbb{E}[\sigma_y]\Big) \quad &\overset{\delta \to 0}{\longrightarrow} \quad |y-x|^{-1/4}, \\ \delta^{-2}\Big(\mathbb{E}\big[\big(\varepsilon_{(x,x+\delta)} - \tfrac{1}{\sqrt{2}}\big)\,\big(\varepsilon_{(y,y+\delta)} - \tfrac{1}{\sqrt{2}}\big)\big]\Big) \quad &\overset{\delta \to 0}{\longrightarrow} \quad |y-x|^{-2}, \end{aligned}$$

where in the powers we can recognize the conformal weights $1/4 = 2h_\sigma + 2\bar{h}_\sigma$ of the spin field σ (with $h_\sigma = \bar{h}_\sigma = 1/16$) and $2 = 2h_\varepsilon + 2\bar{h}_\varepsilon$ of the energy field $\varepsilon_{(x,x+\delta)} := \sigma_x \sigma_{x+\delta}$ (with $h_\varepsilon = \bar{h}_\varepsilon = 1/2$). Let us also record here the fusion rules in the $(4,3)$ minimal model for possible later reference:

$$\begin{aligned} 1 \boxtimes 1 &= 1, \qquad & 1 \boxtimes \sigma &= \sigma, \qquad & 1 \boxtimes \varepsilon &= \varepsilon, \\ \sigma \boxtimes \sigma &= 1 \boxplus \varepsilon, \qquad & \sigma \boxtimes \varepsilon &= \sigma, \qquad & \varepsilon \boxtimes \varepsilon &= 1. \end{aligned}$$

2.3 So, are we done understanding the Ising CFT?

We will see that this is not the end of the story... For example, looking at crossing probabilities in the Ising model, given by an analogue of Cardy's formula, we see that the corresponding correlation functions go outside of the minimal model. (In fact, they present logarithmic features, hinting that the appropriate CFT for the interface observables is a log-CFT.)

[6] There are also loops in the interior of the domain, separating "+" spins and "−" spins. They have been shown to converge to the so-called conformal loop ensemble (CLE(3)) [BH19], a kind of multi-loop version of SLE.

[7] The one with weight $h = 0$ is the trivial module $M_0 \cong \mathbb{C}$ that corresponds to the identity field 1.

3 What is SLE?

Schramm-Loewner evolutions (SLE), originally called "stochastic" Loewner evolutions, were introduced by Schramm [Sch00], who argued that they are the only possible random curves that could describe scaling limits of critical lattice interfaces in 2D systems. Schramm's definition was inspired by the classical theory of Loewner [Loe23] for dynamical description of shrinking domains, encoded in conformal maps. Schramm's revolutionary input was that such maps could also be *random*. One of the first celebrated applications of SLE was the rigorous calculation of critical exponents [LSW01a, LSW01b, SW01, LSW02], in agreement with the earlier predictions in the physics literature [dN83, BPZ84a, BPZ84b, Car84, DF84, DS87, Nie87].

For the basic theory, see the book [Kem17] which also contains the necessary background in stochastic and complex analysis to understand the SLE theory (for mathematicians). For a geometric perspective, see the monograph [Fri04]. See [Car05, BB06] for lectures targeted to theoretical physicists. Here, we try to focus on the intuitive definition and properties.

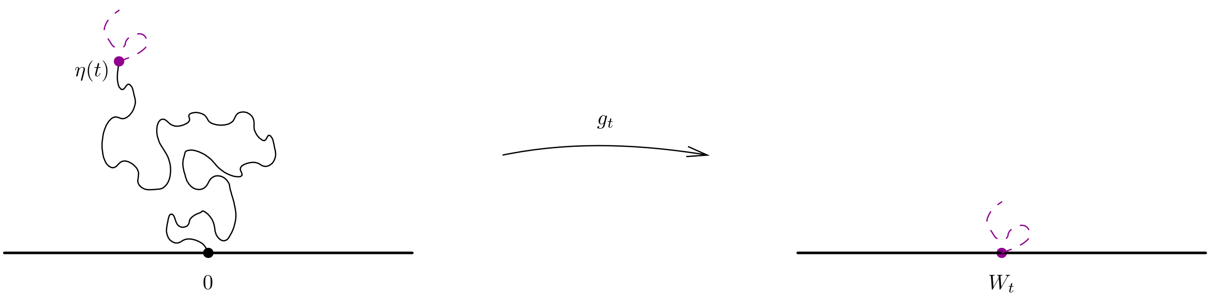


Figure 3.1: Illustration of the Loewner maps $g_t : \mathbb{H} \smallsetminus \eta[0,t] \to \mathbb{H}$ for the SLE(κ) curve η. The image of the tip $\eta(t)$ of the curve is the driving process $W_t = \sqrt{\kappa} B_t$. (Figure from [Pel19].)

3.1 Schramm-Loewner evolution

Precisely speaking, for $\kappa \geq 0$, the (chordal) Schramm-Loewner evolution SLE(κ) is a family of conformally invariant probability measures $\mathbb{P}_{D;x,y}$ on curves, indexed by simply connected domains $D \subsetneq \mathbb{C}$ with two distinct boundary points $x, y \in \partial D$. Each measure[8] $\mathbb{P}_{D;x,y}$ is supported on continuous unparameterized curves in $\overline{D}$ from x to y. By conformal invariance, we can consider the upper half-plane $D = \mathbb{H} := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ as the reference domain (we pick a representative in the moduli space, and we assume the Euclidean metric), and $x = 0$ and $y = \infty$.

Concretely, SLE(κ) curves can be generated dynamically using random Loewner evolutions. This uses some complex analysis and PDE theory, but the idea is simple; see Figure 3.1. In its construction as a growth process, the time evolution of the curve $\eta : [0, \infty) \to \overline{D}$ is encoded in a collection $(g_t)_{t \geq 0}$ of conformal maps $z \mapsto g_t(z)$, which solve an ordinary differential equation[9] in time (Loewner equation):

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}, \qquad g_0(z) = z, \qquad z \in \mathbb{H}, \tag{LE}$$

where $t \mapsto W_t$ is a real-valued continuous function, called the *driving function*. To construct SLE(κ), we take $W_t = \sqrt{\kappa} B_t$, where $(B_t)_{t \geq 0}$ is one-dimensional Brownian motion, that is, a

[8] One can uniquely characterize such a one-parameter family of measures $\mathbb{P}^{D;x,y}$ by conformal invariance and a domain Markov property, which together imply that the Loewner driving function of such a curve must be a constant multiple of one-dimensional Brownian motion. This was proved by Schramm [Sch00].

[9] For each $z \in \mathbb{H}$, this equation is only well-defined up to a blow-up time $\sup\{t > 0 \mid \inf_{s \in [0,t]} |g_s(z) - W_s| > 0\}$, called the *swallowing time* of z. Geometrically, at the swallowing time of a point z the curve either hits the point or forms a bubble around it, disconnecting it from infinity, which results in the time-evolution of that point under the Loewner map g_t to stop. When the speed is small enough, $\kappa \in [0, 4]$, bubbling does not happen almost surely.

random function $B : [0,\infty) \to \mathbb{R}$ describing the time-evolution of the tip of the curve, as in Figure 3.1, with $B_0 = 0$. The map $z \mapsto g_t(z)$ is the unique conformal bijection from $\mathbb{H} \smallsetminus \eta[0,t]$ onto $\mathbb{H}$ with normalization chosen as $\lim_{z\to\infty} |g_t(z) - z| = 0$. See [RS05] for the original reference.

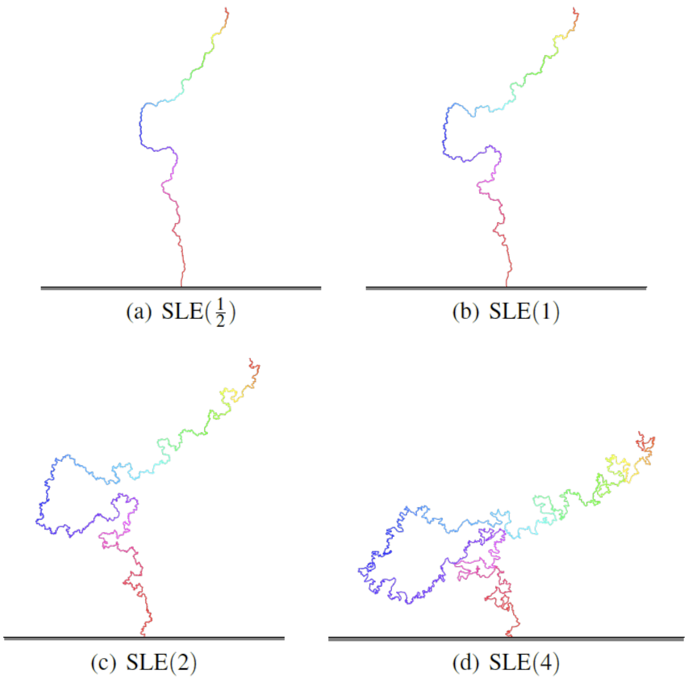


Figure 3.2: Simulations of SLE(κ) curves with different values of κ but the same realization of the driving Brownian motion. (Figure from [Kem17].)

The diffusivity parameter (speed) $\kappa \geq 0$ determines the geometry of the SLE curve (almost surely); see Figure 3.2. For example, when $0 \leq \kappa \leq 4$, the SLE(κ) curve is simple; when $4 < \kappa < 8$, the SLE(κ) curve is not simple, nor space-filling; and when $\kappa \geq 8$, the SLE(κ) curve is space-filling [RS05]. For the purposes of our discussion, we assume throughout that $\kappa \leq 4$.

For later reference (to talk about multiple SLE measures later), we let

$$\mathcal{X}^0(D; x, y)$$

denote the set of continuous simple unparameterized curves in $\overline{D}$ connecting distinct $x \in \partial D$ and $y \in \partial D$ such that they only touch the boundary in $\{x, y\}$. This is the set where the chordal SLE(κ) curves live in our case[10] — that is, when $\kappa \leq 4$ the SLE(κ) probability measure $\mathbb{P}_{D;x,y}$ is supported on $\mathcal{X}^0(D; x, y)$. This (non-compact) space is usually endowed with the metric

$$d_{\mathcal{X}}(\eta, \tilde{\eta}) := \inf_{\psi, \tilde{\psi}} \sup_{t\in[0,1]} |\eta(\psi(t)) - \tilde{\eta}(\tilde{\psi}(t))|,$$

where $\eta, \tilde{\eta} : [0,1] \to \overline{D}$ are representatives of curves, and the infimum is taken over all reparameterizations, that is, increasing bijections $\psi, \tilde{\psi} : [0,1] \to [0,1]$.

3.2 Conformal anomaly

The SLE(κ) curve model is a conformally invariant model. Moreover, it is closely related to CFT — and in fact, as we will see, the interaction of SLE(κ) curves can be encoded in correlation functions of certain primary fields. But where is the central charge and the $\mathfrak{Vir}$-action?

[10] Here we also need the *reversibility property* of chordal SLE: the SLE(κ) curve from x to y has the same distribution as the SLE(κ) curve from y to x. This property is very non-trivial to prove [Zha08].

The anomaly[11] can be detected by chopping off a subset $A \subset \overline{D}$ from our domain and looking how the probability measure of the SLE(κ) curve changes. Let us consider concretely a setup where $D = \mathbb{D}$ and $\mathbb{D} \smallsetminus A =: U$ is simply connected and contains the endpoints $x, y \in \partial U \cap \partial \mathbb{D}$ of the curve. Let $f_A : \mathbb{D} \smallsetminus A \to \mathbb{D}$ be the unique conformal bijection such that $f_A(x) = x$ and $f_A(y) = y$.

In measure-theoretic terms, chordal SLE(κ) measure in $(U; x, y)$ is absolutely continuous with respect to chordal SLE(κ) measure in $(\mathbb{D}; x, y)$, with Radon-Nikodym derivative given by

$$\frac{\mathrm{d}\mathbb{P}^{U;x,y}}{\mathrm{d}\mathbb{P}^{\mathbb{D};x,y}}(\eta) = \mathbb{1}\{\eta \cap A = \varnothing\} \, \frac{\exp\left(\frac{c}{2}\, \mu_{\mathbb{D}}^{\mathrm{loop}}(\eta, A)\right)}{|f_A'(x)|^{h_{1,2}}\, |f_A'(y)|^{h_{1,2}}}, \qquad U = \mathbb{D} \smallsetminus A, \tag{3.1}$$

where $\mathbb{1}\{\eta \cap A = \varnothing\}$ is the indicator function on the space of curves for the event $\{\eta \cap A = \varnothing\}$ that the curve avoids the set A, i.e.,

$$\mathbb{1}\{\eta \cap A = \varnothing\} = \begin{cases} 0, & \eta \cap A \neq \varnothing, \\ 1, & \eta \cap A = \varnothing, \end{cases}$$

and where $\mu_{\mathbb{D}}^{\mathrm{loop}}(\eta, A)$ is a conformal invariant, to be discussed shortly, and

$$c = c(\kappa) = \frac{(3\kappa - 8)(6 - \kappa)}{2\kappa}$$

is the central charge of the corresponding CFT, also written in the more familiar form (1.5):

$$c = 13 - 6(\theta + \theta^{-1}), \qquad \theta = \frac{\kappa}{4},$$

and $h_{1,2}$ is a conformal weight in the Kac table (1.5):

$$h_{1,2} = h_{1,2}(\kappa) = \frac{6 - \kappa}{2\kappa}.$$

Recall from the representation theory of $\mathfrak{Vir}$ (Theorem 1.1 in Section 1) that the Verma module with central charge c and conformal weight h possesses nontrivial submodules if and only if h belongs to the Kac table:

$$h_{r,s} = \frac{(r^2 - 1)}{4}\theta + \frac{(s^2 - 1)}{4}\theta^{-1} + \frac{(1 - rs)}{2}, \qquad \theta = \frac{\kappa}{4},$$

for $r, s \in \mathbb{N}$. (Of course, the formula makes sense for more general r, s and we will use it below.)

The conformally invariant object $\mu_D^{\mathrm{loop}}(\eta, A)$ can be written in terms of the *Brownian loop measure* μ_D^{loop} introduced by Lawler, Schramm & Werner [LSW03, LW04], which is an infinite measure on collections of 2D Brownian motion loops in D. We denote by $\mu_D^{\mathrm{loop}}(\eta, A)$ the measure of those Brownian loops in D that intersect both sets η and A. It can be proven to be finite when $A \cap \eta = \varnothing$. A potentially more intuitive formula can be written as[12]

$$\mu_D^{\mathrm{loop}}(\eta, A) \;=\; \log\left(\frac{\det_\zeta(\Delta_{\mathrm{int}(A)})\, \det_\zeta(\Delta_{D \smallsetminus (\eta \cup A)})}{\det_\zeta(\Delta_D)}\right) \;=\; \log\left(\frac{\det_\zeta(\Delta_{D \smallsetminus A})\, \det_\zeta(\Delta_{D \smallsetminus \eta})}{\det_\zeta(\Delta_D)\, \det_\zeta(\Delta_{(D \smallsetminus A) \smallsetminus \eta})}\right)$$

where Δ_V is the Laplace-Beltrami operator on the domain V with Dirichlet boundary conditions, and $\det_\zeta$ is its ζ-regularized determinant. A caveat here is that the right-hand side formula is *only* guaranteed to be finite when $A \cap \eta = \varnothing$ and both ∂A and η are smooth enough. (*But* SLE(κ) curves are not smooth at all — they are fractal with non-trivial Hausdorff dimension.)

[11] In fact, one should view this as a conformal change of metric, which gives rise to the Weyl anomaly.

[12] Note that the domain $D \smallsetminus \eta$ has two components, and the Laplacian determinant is understood just as the product of the ones in each component. Similarly for $\mathrm{int}(A)$ if it has several components.

To see the conformal anomaly in (3.1) in terms of a change of metric, taking $D = \mathbb{D}$ with the flat metric g_0, the above heuristic formula becomes

$$\begin{aligned}\mu_D^{\mathrm{loop}}(\eta, \mathbb{D} \smallsetminus U) &= \log \left(\frac{\det_\zeta(\Delta_{U,\mathrm{g}_0}) \det_\zeta(\Delta_{\mathbb{D}\smallsetminus\eta,\mathrm{g}_0})}{\det_\zeta(\Delta_{\mathbb{D},\mathrm{g}_0}) \det_\zeta(\Delta_{U\smallsetminus\eta,\mathrm{g}_0})} \right) \\ &= \log \left(\frac{\det_\zeta(\Delta_{\mathbb{D},|(f_A^{-1})'|^2\mathrm{g}_0}) \det_\zeta(\Delta_{\mathbb{D}\smallsetminus\eta,\mathrm{g}_0})}{\det_\zeta(\Delta_{\mathbb{D},\mathrm{g}_0}) \det_\zeta(\Delta_{\mathbb{D}\smallsetminus f_A^{-1}(\eta)),|(f_A^{-1})'|^2\mathrm{g}_0})} \right), \qquad U = \mathbb{D} \smallsetminus A,\end{aligned}$$

where the uniformizing map $f_A : U \to \mathbb{D}$ pulls back the flat metric from $\mathbb{D}$ to U as $|(f_A^{-1})'|^2\mathrm{g}_0$.

Let us remark that on a domain D with metric g and smooth boundary (above, we have been using the flat metric and omitted it from the notation),

$$\left(\frac{\det{}_\zeta(\Delta_{D,\mathrm{g}})}{\exp\left(\frac{1}{4\pi}\int_{\partial\Sigma} k_\mathrm{g}\widetilde{\mathrm{vol}}_\mathrm{g}\right)} \right)^{-c/2} = Z_\mathrm{g}(D)$$

can be seen as a partition function (with $c = 1$, arguably of the free boson). Any CFT partition function on surface Σ transforms as

$$Z_{e^{2\sigma}\mathrm{g}}(\Sigma) \;=\; e^{c\,S(\sigma,\mathrm{g})} Z_\mathrm{g}(\Sigma)$$

where two metrics g and $e^{2\sigma}\mathrm{g}$ in the same conformal class are related by the anomaly functional

$$S(\sigma,\mathrm{g}) \;:=\; \frac{1}{12\pi}\iint_\Sigma \left(\frac{1}{2}|\nabla_\mathrm{g}\sigma|_\mathrm{g}^2 + R_\mathrm{g}\sigma\right)\mathrm{vol}_g \;+\; \frac{1}{12\pi}\int_{\partial\Sigma} k_\mathrm{g}\sigma\,\widetilde{\mathrm{vol}}_\mathrm{g}, \qquad \sigma \in C^\infty(\Sigma,\mathbb{R}),$$

where ∇_g, R_g, vol_g, k_g, $\widetilde{\mathrm{vol}}_\mathrm{g}$ are respectively the divergence, Gaussian curvature, and volume form on Σ, and the boundary curvature and volume form on $\partial\Sigma$, induced by g. For the Laplacian determinant, this is related to Polyakov-Alvarez conformal anomaly formula [Pol81, Alv83].

3.3 Comments of the radial case

Let us also briefly mention another SLE interesting variant here. Radial SLE(κ) is a curve growing from a boundary point, say $1 \in \partial\mathbb{D} := \{z \in \mathbb{C} \mid |z| = 1\}$, to an interior point, say $0 \in \mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}$, in a simply connected domain, say $\mathbb{D}$. It can be generated by using a radial version of Loewner equation (which we will not need).

Consider $A \subset \overline{\mathbb{D}}$ such that $\mathbb{D} \smallsetminus A =: U$ is simply connected and contains the endpoints 0 and 1 of the curve. Let $f_A : \mathbb{D} \smallsetminus A \to \mathbb{D}$ be the unique conformal bijection such that $f_A(0) = 0$ and $f_A(1) = 1$. Then, radial SLE(κ) measure in $\mathbb{D} \smallsetminus A$ from 1 to 0 is absolutely continuous with respect to radial SLE(κ) measure in $\mathbb{D}$ from 1 to 0, with Radon-Nikodym derivative given by

$$\frac{\mathrm{d}\mathbb{P}^{\mathbb{D}\smallsetminus A;1,0}}{\mathrm{d}\mathbb{P}^{\mathbb{D};1,0}}(\eta) = \mathbb{1}\{\eta \cap A = \varnothing\} \; \frac{\exp\left(\frac{c}{2}\,\mu_\mathbb{D}^{\mathrm{loop}}(\eta, A)\right)}{|f_A'(0)|^{2h_{0,1/2}}\,|f_A'(1)|^{h_{1,2}}},$$

where

$$h_{0,1/2} = h_{0,1/2}(\kappa) = \frac{(6-\kappa)(\kappa-2)}{16\kappa}.$$

The fact that the weight at the interior point is $2h_{0,1/2}$ is because it is a bulk field, whereas the boundary endpoints are boundary fields, which could also be regarded as chiral fields.

Note that this is consistent with the predictions from loop models and corresponds to an *electric vertex operator* at the origin. One can also add a *magnetic charge*, which results in spiraling behavior for the curve around the origin. See [HPW25] and references therein.

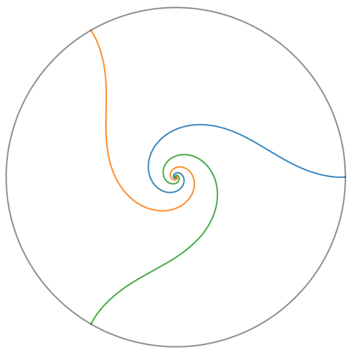


Figure 3.3: Illustration for the semiclassical limit of 3-radial spiraling SLE. (Figure by Mo Chen.)

Interestingly enough, if one takes the limit where the interior endpoint of the curve tends to the boundary, one recovers the above chordal case — perhaps surprisingly, this also holds with the magnetic charge and multiple curves. Namely, if we consider n curves started at boundary points $x_1, \ldots, x_n \in \partial\mathbb{D}$ and growing towards the origin, the associated conformal block is that involving the familiar fields $\phi_{1,2}(x_1), \ldots, \phi_{1,2}(x_n)$ and a bulk field at the origin of weights $(h, \bar{h})$,

$$h = h_{0,n/2} - \frac{\mu^2}{4\kappa} - \mathrm{i}\frac{\mu}{2\kappa}n, \qquad \bar{h} = h_{0,n/2} - \frac{\mu^2}{4\kappa} + \mathrm{i}\frac{\mu}{2\kappa}n,$$

where $\mu \in \mathbb{R}$ is the spiraling rate. Somewhat mysteriously, in the limit where the interior endpoint tends to the boundary, the associated bulk field (which has nonzero spin and is nondegenerate) becomes a boundary field of weight $h_{1,n+1}$ (which is degenerate at level $n+1$) [HPW25].

4 Correlation functions

In this section, we will consider a collection of chordal/boundary conformal blocks $\{\mathcal{Z}_\alpha\}_\alpha$ related to the SLE(κ) curve model. For $\kappa \in (0, 8]$ they are linearly independent functions that span a space of solutions to the level two BPZ equations, and could thus be regarded as *conformal blocks* of the fields $\phi_{1,2}$. One could also compute fusion rules of these conformal blocks, and see that one gets the complete first row of the Kac table. See [Pel19, FLPW24] and references therein. In particular, these blocks are beyond the Kac table! (In fact, at rational values of κ, they also reveal logarithmic behavior; the easiest example being $\kappa = 8$ with $c = -2$ [LPW24].)

From these chiral blocks, one can use the crossing symmetry idea, common in the classical CFT literature [DF85, DFMS97] to construct single-valued bulk correlation functions in a diagonal theory. This has been only partly written up, but the program is explained in [Pel16]. The proof relies on a hidden quantum group symmetry and quantum Schur-Weyl duality developed in [KP16, KP20, FP20, Pel20, FP]. (See citations therein for background references.)

4.1 Back to critical interfaces

Let $D \subsetneq \mathbb{C}$ be a simply connected domain with $2N$ distinct marked points $x_1, x_2, \ldots, x_{2N} \in \partial D$ appearing in counterclockwise order along the boundary. Like discussed earlier, by imposing alternating "+" and "−" boundary conditions to the Ising model changing at each insertion x_j, we see a family of interacting chordal interfaces $\overline{\gamma}^\delta = (\gamma_1^\delta, \ldots, \gamma_N^\delta)$ in the model. These interfaces converge [Smi10, CDCH+14, Izy15, BPW21] in the scaling limit $\delta \to 0$ to the multiple (chordal)

N-SLE(κ) with $\kappa = 3$, discussed below. (We noted earlier that the parameter $\kappa = 3$ matches with the central charge $c = 1/2$, and it is the only such κ that gives a simple curve.)

These curves can have various planar connectivities that we label by planar link patterns

$$\alpha = \{\{a_1, b_1\}, \ldots, \{a_N, b_N\}\} \in \mathrm{LP}_N, \tag{4.1}$$

where $\{a_1, b_1, \ldots, a_N, b_N\} = \{1, 2, \ldots, 2N\}$. Note that for each fixed $N \in \mathbb{N}$, the total number of planar link patterns is the *Catalan number* $C_N = \frac{1}{N+1}\binom{2N}{N} = \#\mathrm{LP}_N$.

We proved in [PW23] in general that the following convergence holds:

$$\lim_{\delta\to 0} \mathbb{P}^\delta[\text{ interfaces } (\gamma_1^\delta, \ldots, \gamma_N^\delta) \text{ form connectivity } \alpha\,] = \frac{\mathcal{Z}_\alpha(D; x_1, \ldots, x_{2N})}{\sum_{\beta\in\mathrm{LP}_N} \mathcal{Z}_\beta(D; x_1, \ldots, x_{2N})}, \tag{4.2}$$

where $\{\mathcal{Z}_\alpha \mid \alpha \in \mathrm{LP}_N\}$ are the "pure partition functions" of multiple chordal SLE(κ) with $\kappa = 3$, to be discussed shortly. Here, the denominator is just a normalization factor that makes the sum of all probabilities 1 (though in other models, it becomes much more interesting [LPW24, FPW24].)

In the first nontrivial case of $N = 2$, the crossing formula (4.2) was predicted in [ASA02], and it is an analogue of Cardy's formula [Car92] for critical Bernoulli percolation: the pure partition functions in this case are given by

$$\mathcal{Z}_{\overset{\frown}{\frown}}(\mathbb{H}; x_1, x_2, x_3, x_4) = \frac{2\Gamma(4/3)}{\Gamma(8/3)\Gamma(5/3)} \left(\frac{(x_2 - x_1)(x_4 - x_3)}{(x_4 - x_2)(x_3 - x_1)}\right)^{2/3} \frac{{}_2F_1\big(\frac{4}{3}, -\frac{1}{3}, \frac{8}{3}; \frac{(x_2-x_1)(x_4-x_3)}{(x_4-x_2)(x_3-x_1)}\big)}{(x_4 - x_1)(x_3 - x_2)},$$
$$\mathcal{Z}_{\frown\frown}(\mathbb{H}; x_1, x_2, x_3, x_4) = \frac{2\Gamma(4/3)}{\Gamma(8/3)\Gamma(5/3)} \left(\frac{(x_4 - x_1)(x_3 - x_2)}{(x_4 - x_2)(x_3 - x_1)}\right)^{2/3} \frac{{}_2F_1\big(\frac{4}{3}, -\frac{1}{3}, \frac{8}{3}; \frac{(x_4-x_1)(x_3-x_2)}{(x_4-x_2)(x_3-x_1)}\big)}{(x_2 - x_1)(x_4 - x_3)},$$

for $x_1 < x_2 < x_3 < x_4$, where $\frown\frown = \{\{1,2\},\{3,4\}\}$ and $\overset{\frown}{\frown} = \{\{1,4\},\{2,3\}\}$ are the two possible link patterns. These formulas and certain other special cases appear in [BBK05, Izy15] (and also in the PhD thesis of Izyurov, who proved the convergence in the $N = 2$ case). In general, explicit formulas for the probability amplitudes $\mathcal{Z}_\alpha$ are not known when $\kappa = 3$.

Let us now compare this with the Ising minimal CFT. The conformal weight $h_{1,2}$ apprearing in the conformal anomaly formula reads

$$h_{1,2} = \frac{1}{2} \qquad (\kappa = 3),$$

which we recognize as the weight of the energy field ε. Take $x_1 = 0$, $x_2 = z$, $x_3 = 1$, $x_4 = \infty$. Then we have

$$\mathcal{Z}_{\overset{\frown}{\frown}}(z) = \frac{2\Gamma(4/3)}{\Gamma(5/3)\Gamma(8/3)}\, z^{2/3}(1 - z)^{-1}\, {}_2F_1\big(\tfrac{4}{\kappa}, 1 - \tfrac{4}{\kappa}, \tfrac{8}{\kappa}; z\big),$$
$$\mathcal{Z}_{\frown\frown}(z) = \frac{2\Gamma(4/3)}{\Gamma(5/3)\Gamma(8/3)}\, (1 - z)^{2/3} z^{-1}\, {}_2F_1\big(\tfrac{4}{\kappa}, 1 - \tfrac{4}{\kappa}, \tfrac{8}{\kappa}; 1 - z\big),$$

and

$$\mathcal{Z}_{\overset{\frown}{\frown}}(z) + \mathcal{Z}_{\frown\frown}(z) = \frac{z^2 - z + 1}{z(1 - z)} = z^{2/3}(1 - z)^{2/3}\, I_2(z),$$

where $I_2(z) = \langle \varepsilon(0)\, \varepsilon(z)\, \varepsilon(1)\, \varepsilon(\infty)\rangle$ is the minimal model chiral conformal block appearing in the classical CFT literature [DF85, DFMS97]. We see that it decomposes further into two functions, which could be viewed as chiral conformal blocks in a larger theory.

4.2 Multiple interacting SLE curves

In the continuum model, we consider curves $\overline{\gamma} = (\gamma_1, \ldots, \gamma_N)$ in D each of which connects two points among $\{x_1, x_2, \ldots, x_{2N}\}$. Let us assume that $\kappa \in (0,4]$, so that the SLE(κ) curves are simple. For each link pattern $\alpha \in \mathrm{LP}_N$, let $\mathcal{X}_\alpha(D; x_1, \ldots, x_{2N})$ denote the set of families $\overline{\gamma} = (\gamma_1, \ldots, \gamma_N)$ of pairwise disjoint curves such that $\gamma_j \in \mathcal{X}^0(D; x_{a_j}, x_{b_j})$ for all $j \in \{1, 2, \ldots, N\}$.

For $N \geq 2$ and $\alpha \in \mathrm{LP}_N$, we define a (chordal) N-SLE(κ) *associated to* α as the unique probability measure on the curve families $\overline{\gamma} \in \mathcal{X}_\alpha(D; x_1, \ldots, x_{2N})$ with the following "resampling" property: *For each $j \in \{1, 2, \ldots, N\}$, the conditional distribution of the curve γ_j given $\{\gamma_1, \gamma_2, \ldots, \gamma_N\} \smallsetminus \{\gamma_j\}$ is the chordal SLE(κ) connecting x_{a_j} and x_{b_j} in the connected component of the domain $D \smallsetminus \bigcup_{i \neq j} \gamma_i$ containing the endpoints x_{a_j} and x_{b_j} of γ_j on its boundary.*

Multichordal SLE(κ) is a family of SLE(κ) curves with interaction. Looking at Figure 2.1 depicting the Ising interfaces, the idea becomes clear: If one discovers part of the model, then the remaining model is just a similar model with fewer curves. The defining property of N-SLE(κ) also implies that for any subset $J \subsetneq \{1, 2, \ldots, N\}$, the conditional distribution of the curves $(\gamma_j)_{j \in J}$ given $\{(\gamma_j)_{j \notin J}\}$ is the appropriate multichordal SLE(κ) (in a potentially disconnected domain, but we can obviously define the random curves in each component separately).

The resampling property can be understood in terms of the conformal anomaly. Indeed, suppose we discovered the curves $\{\gamma_2, \gamma_3, \ldots, \gamma_N\}$ — meaning we condition on them in the multiple SLE(κ) probability measure. Then, with $D = \mathbb{H}$ as before, the appropriate component of the domain $\mathbb{H} \smallsetminus \{\gamma_2, \gamma_3, \ldots, \gamma_N\}$ containing the endpoints x_{a_1} and x_{b_1} of γ_1 on its boundary can be viewed as $\mathbb{H} \smallsetminus A$, where A is formed by the components chopped off by the other curves.

With this idea in mind, multichordal SLE(κ) with central charge $c = c(\kappa)$ can be constructed as follows [Law09]. Let P^κ_α denote the product measure of N independent chordal SLE(κ) curves associated to the link pattern α. Denote by E^κ_α the expectation with respect to P^κ_α. The N-SLE(κ) probability measure $\mathbb{P}^\kappa_\alpha$ on $\mathcal{X}_\alpha(D; x_1, \ldots, x_{2N})$ can be obtained by weighting P^κ_α with the Radon-Nikodym derivative[13] (which is a measurable map on the curve space)

$$R^\kappa_\alpha(\overline{\gamma}) = \frac{\mathrm{d}\mathbb{P}^\kappa_\alpha}{\mathrm{d}\mathsf{P}^\kappa_\alpha}(\overline{\gamma}) := \frac{\mathbb{1}\{\gamma_i \cap \gamma_j = \varnothing \ \forall i \neq j\} \ \exp\left(\frac{c}{2} m_D(\overline{\gamma})\right)}{\mathsf{E}^\kappa_\alpha\left[\mathbb{1}\{\gamma_i \cap \gamma_j = \varnothing \ \forall i \neq j\} \ \exp\left(\frac{c}{2} m_D(\overline{\gamma})\right)\right]}, \tag{4.3}$$

where $m_D(\overline{\gamma})$ is expressed in terms of the Brownian loop measure: $m_D(\gamma) := 0$ if $N = 1$, and

$$m_D(\overline{\gamma}) := \sum_{p=2}^{N} \mu^{\mathrm{loop}}_D\Big(\big\{\ell \mid \ell \cap \gamma_i \neq \varnothing \text{ for at least } p \text{ of the } i \in \{1, \ldots, N\}\big\}\Big) \qquad \text{if } N \geq 2.$$

One can prove that this is a conformally invariant quantity. See [Law09, PW19] for more details.

The *pure partition functions* of multiple SLE(κ) are defined in terms of the total mass of the N-SLE(κ) measure (here, we use the labeling (4.1) of the curve endpoints):

$$\mathcal{Z}_\alpha(D; x_1, \ldots, x_{2N}) := \left(\prod_{j=1}^{N} H_D(x_{a_j}, x_{b_j})\right)^{h_{1,2}} \mathsf{E}^\kappa_\alpha\Big[\mathbb{1}\{\gamma_i \cap \gamma_j = \varnothing \ \forall i \neq j\} \ \exp\Big(\frac{c}{2} m_D(\overline{\gamma})\Big)\Big], \tag{4.4}$$

where $H_D(x, y)$ is the boundary Poisson kernel, that is, the unique conformally covariant function defined as

$$H_D(x, y) := |\varphi'(x)| \, |\varphi'(y)| \, H_{\mathbb{H}}(\varphi(x), \varphi(y)), \qquad \text{where} \qquad H_{\mathbb{H}}(z, w) := |w - z|^{-2},$$

where $\varphi : D \to \mathbb{H}$ is any conformal map. Note that $\mathcal{Z}_\alpha(D; x_1, \ldots, x_{2N})$ is a function of the boundary points $x_1, \ldots, x_{2N}$ and the domain D, as well as of the link pattern α. (In fact, in general it should be a function of the associated conformal moduli in the problem.)

[13] The difference of 1/2 compared to [BPW21] is due to normalization conventions for Brownian loop measure.

The motivation for this definition comes from considering chordal SLE(κ) as a unnormalized (non-probability) measure on $\mathcal{X}^0(\mathbb{H}; x_{a_j}, x_{b_j})$, which has the correct conformal anomaly:

$$\frac{\mathrm{d}\mathbb{P}^{\mathbb{H}\smallsetminus A; x_{a_j}, x_{b_j}}}{\mathrm{d}\mathbb{P}^{\mathbb{H}; x_{a_j}, x_{b_j}}}(\eta) = \mathbb{1}\{\gamma_j \cap A = \varnothing\} \, \frac{\exp\big(\frac{c}{2}\, \mu_{\mathbb{H}}^{\mathrm{loop}}(\gamma_j, A)\big)}{|f_A'(x_{a_j})|^{h_{1,2}}\, |f_A'(x_{b_j})|^{h_{1,2}}},$$

Indeed, using the definition of the Poisson kernel, we obtain

$$\begin{aligned} &\mathbb{1}\{\gamma_j \cap A = \varnothing\} \, \frac{\exp\big(\frac{c}{2}\, \mu_{\mathbb{H}}^{\mathrm{loop}}(\gamma_j, A)\big)}{|f_A'(x_{a_j})|^{h_{1,2}}\, |f_A'(x_{b_j})|^{h_{1,2}}} \\ &= \mathbb{1}\{\gamma \cap A = \varnothing\} \left(\frac{H_{\mathbb{H}}(x_{a_j}, x_{b_j})}{H_{\mathbb{H}\smallsetminus A}(x_{a_j}, x_{b_j})}\right)^{h_{1,2}} \exp\Big(\frac{c}{2}\, \mu_{\mathbb{H}}^{\mathrm{loop}}(\gamma_j, A)\Big) \end{aligned}$$

and rearranging this, we obtain

$$\big(H_{\mathbb{H}\smallsetminus A}(x_{a_j}, x_{b_j})\big)^{h_{1,2}} \, \mathrm{d}\mathbb{P}^{\mathbb{H}\smallsetminus A; x_{a_j}, x_{b_j}}(\gamma) = \big(H_{\mathbb{H}}(x_{a_j}, x_{b_j})\big)^{h_{1,2}} \, \exp\Big(\frac{c}{2}\, \mu_{\mathbb{H}}^{\mathrm{loop}}(\gamma_j, A)\Big) \, \mathrm{d}\mathbb{P}^{\mathbb{H}; x_{a_j}, x_{b_j}}(\gamma).$$

The total mass of the right-hand side appears as an ingredient in the construction of $\mathcal{Z}_\alpha$. An inclusion-exclusion argument gives the loop measure term $m_D(\overline{\gamma})$, see [Law09, PW19, BPW21].

4.3 Some heuristics on the SLE action functional

We can also write the Radon-Nikodym derivative (4.3) in the form

$$R_\alpha^\kappa(\overline{\gamma}) = \frac{\mathbb{1}(\overline{\gamma}) \, \exp\Big(\frac{1}{\kappa}\, \Phi_\kappa(\overline{\gamma})\Big)}{\mathsf{E}_\alpha^\kappa\Big[\mathbb{1}(\overline{\gamma}) \, \exp\Big(\frac{1}{\kappa}\, \Phi_\kappa(\overline{\gamma})\Big)\Big]}, \qquad \Phi_\kappa(\overline{\gamma}) := \kappa \, \frac{c(\kappa)}{2} \, m_D(\overline{\gamma}),$$

writing $\mathbb{1}(\overline{\gamma}) = \mathbb{1}\{\gamma_i \cap \gamma_j = \varnothing \; \forall i \neq j\}$ for the indicator function that the curves are disjoint.

This looks a sort of "Bolzmann measure on curves" — and one could think of the quantity $\Phi_\kappa(\overline{\gamma})$ as an "action functional" for SLEs (though there are also other terms that it could include, obtained from "Loewner energies" of the curves as well as terms involving the conformal moduli of the problem, in this case, Poisson kernels as above; see [PW24, Eq. (1.4)].)

Note that $\frac{\kappa\, c(\kappa)}{2} \to -24$ as $\kappa \to 0$. In fact, in this semiclassical limit $c(\kappa) \to -\infty$, the measure concentrates on "interacting geodesics" in the hyperbolic metric on D (see [PW24]), meaning curve families $\overline{\eta} = (\eta_1, \ldots, \eta_N)$ having the following property inherited from the resampling:

For each $j \in \{1, 2, \ldots, N\}$, the curve η_j is the hyperbolic geodesic[14]*, that is, the chordal SLE(0), between the points x_{a_j}, x_{b_j} in the connected component of $D \smallsetminus \bigcup_{i \neq j} \eta_i$ containing η_j.*

4.4 BPZ equations from SLE martingales

Bauer and Bernard observed [BB03, BBK05] that the time-evolution of an SLE(κ) curve gives rise to the BPZ PDE at level two, using a martingale argument from probability theory[15] that we now sketch. (This was later elaborated especially by Dubédat [Dub07, Dub06, KP16].) The rigorous arguments involve probability theory and analysis, but the conceptual idea is simple.

For us, let's take as a black box the fact that by a standard result in probability theory known as Girsanov's theorem (see, e.g. [RY05, Chapter 8]) the change of measure

$$R_\alpha^\kappa(\overline{\gamma}) = \frac{\exp\Big(\frac{1}{\kappa}\, \Phi_\kappa(\overline{\gamma})\Big)}{\mathsf{E}_\alpha^\kappa\Big[\exp\Big(\frac{1}{\kappa}\, \Phi_\kappa(\overline{\gamma})\Big)\Big]}, \qquad \Phi_\kappa(\overline{\gamma}) := \kappa \, \frac{c(\kappa)}{2} \, m_D(\overline{\gamma}),$$

[14] A *hyperbolic geodesic* in $(D; x, y)$ is the image of $[-1, 1]$ by a conformal map $(\mathbb{D}; 1, -1) \to (D; x, y)$.

[15] Martingales are processes $(M_t)_{t \geq 0}$ such that, given the history up to time t, the conditional expectation of M observed at time $s \geq t$ equals the present value M_t. So in a sense, they don't remember their past.

is obtained by changing the measure of the Brownian motion B that appears in the Loewner evolution (LE) of the SLE(κ) curve. In the language of stochastic analysis, the curves in a multiple SLE(κ) can be described via a Loewner evolution similar to the usual chordal case (LE), but where the Loewner driving process W_t (which for one curve was Brownian motion with speed κ) has a *drift* given by the *interaction* with the other curves. We can recursively focus on growing one curve at a time (marginal law), so that the endpoints of the other curves serve as spectator points with respect to the evolution of the curve that we are growing at the time.

It turns out that one can also write the drift for the driving function in a quite convenient form [Dub07]. On the upper half-plane $\mathbb{H}$ with marked points $x_1 < \cdots < x_{2N}$, for the marginal law of the curve starting from x_j, with $j \in \{1, \ldots, 2N\}$, we have the SDEs

$$\begin{cases} \mathrm{d}W_t = \sqrt{\kappa}\ \mathrm{d}B_t + \kappa\, \partial_j \log \mathcal{Z}_\alpha\big(g_t(x_1), \ldots, g_t(x_{j-1}), W_t, g_t(x_{j+1}), \ldots, g_t(x_{2N})\big)\ \mathrm{d}t, \\ \mathrm{d}g_t(x_i) = \frac{2\ \mathrm{d}t}{g_t(x_i) - W_t}, \qquad \text{for } i \neq j, \end{cases} \tag{4.5}$$

with initial conditions

$$\begin{cases} W_0 = x_j, \\ g_0(x_i) = x_i, \qquad \text{for } i \neq j. \end{cases}$$

From the point of view of Girsanov's theorem, the drift $(\kappa\, \partial_j \log \mathcal{Z}_\alpha)\ \mathrm{d}t$ arises from a martingale that encodes the interaction (see [Dub07] for the derivation):

$$M_t \ = \ \prod_{i \neq j} |g_t'(x_i)|^{h_{1,2}} \times \mathcal{Z}\big(g_t(x_1), \ldots, g_t(x_{j-1}), \sqrt{\kappa} B_t + x_j, g_t(x_{j+1}), \ldots, g_t(x_{2N})\big),$$

where g_t is the solution to the Loewner equation (LE) with driving function $W_t = \sqrt{\kappa} B_t + x_j$.

It is straightforward to formally calculate the Itô (stochastic) differential of this martingale using Itô's formula (see e.g. [RW00, Theorem (32.8)]), the observation $g_t'(z) > 0$, and the relations

$$\mathrm{d}g_t(z) = \frac{2}{g_t(z) - W_t}\ \mathrm{d}t \qquad \text{and} \qquad \mathrm{d}g_t'(z) = -\frac{2 g_t'(z)}{(g_t(z) - W_t)^2}\ \mathrm{d}t,$$

which follow from the Loewner equation (LE). By the martingale property, the drift term in the result should equal zero, which gives the following second order PDE (compare with (1.7)):

$$\left[\frac{\kappa}{2} \frac{\partial^2}{\partial x_j^2} + \sum_{i \neq j} \left(\frac{2}{x_i - x_j} \frac{\partial}{\partial x_i} - \frac{2 h_{1,2}(\kappa)}{(x_i - x_j)^2} \right)\right] \mathcal{Z}_\alpha(x_1, \ldots, x_{2N}) = 0. \tag{4.6}$$

This equation holds symmetrically for all $j \in \{1, \ldots, 2N\}$ [Dub07], since it does not matter which curve we start growing first, and it only involves studying the evolution at small times.

This is (almost) a rigorous derivation of the level two BPZ PDE for the multichordal SLE(κ) system. What remains is to show that the function $\mathcal{Z}_\alpha$ is regular enough that one can complete the argument involving stochastic differentiation. This can be done in at least two ways:

- ▷ Show that $\mathcal{Z}_\alpha$ is C^2, and then one can apply Itô's formula [JL18].
- ▷ Show that $\mathcal{Z}_\alpha$ is continuous, use this to argue that one gets a weak (distributional) solution to (4.6), and then invoke the fact that the PDE is hypoelliptic to conclude that in fact, $\mathcal{Z}_\alpha$ is also a strong solution, in particular smooth; see [Dub15] and [FLPW24, Appendix B].

One can generalize the above argument to include other marked points which have other conformal weights, as in (1.7). What is important is that the point x_j where the curve is growing from corresponds to the Kac conformal weight $h_{1,2}$.

The content of the Pascal Institute lectures ends here.

5 Semiclassical limit

When κ is small, the SLE(κ) curves tend to be relatively straight, but still fractal — see Figures 3.1 & 3.3. In the limit $\kappa \to 0$, the SLE(κ) probability measure on the curve space $\mathcal{X}^0(D; x, y)$ concentrates on an atomic measure supported on the hyperbolic geodesic connecting the two endpoints of the curve — that is, the curve becomes the shortest path connecting its endpoints. To formulate this precisely, probabilists talk about a large deviation principle (LDP).

5.1 Large deviation principle

Intuitively, for a given reference curve η in $\overline{D}$ connecting two boundary points $x, y \in \partial D$, when the speed κ of the driving Brownian motion goes to zero, we expect a limiting behavior of type

$$\text{“}\,\mathbb{P}[\text{SLE}(\kappa) \text{ curve } \gamma \text{ in } (D; x, y) \text{ stays close to } \eta] \quad \overset{\kappa \to 0+}{\approx} \quad \exp\Big(- \frac{I_{D;x,y}(\eta)}{\kappa} \Big)\text{”},$$

where $I_{D;x,y}$ is a conformally invariant quantity called *Loewner energy* of the curve η:

$$I_{D;x,y}(\eta) := I_{\mathbb{H};0,\infty}(\varphi(\eta)) := \frac{1}{2} \int_0^\infty |W'(t)|^2 \, \mathrm{d}t,$$

and where W is the Loewner driving function of the reference curve $\varphi(\eta)$, that is the image of η under any conformal map $\varphi \colon D \to \mathbb{H}$ sending x and y respectively to 0 and ∞. (The right-hand side is just the Dirichlet energy of W.) The Loewner energy of a chord is not always finite.

Similarly, for multiple curves, taking a reference curve family $\overline{\eta} = (\eta_1, \ldots, \eta_N)$, we have

$$\text{“}\,\mathbb{P}[N\text{-SLE}(\kappa) \text{ curves } \overline{\gamma} \text{ in } (D; x_1, \ldots, x_{2N}) \text{ stay close to } \overline{\eta}] \quad \overset{\kappa \to 0+}{\approx} \quad \exp\Big(- \frac{I_{D;x_1,\ldots,x_{2N}}(\overline{\eta})}{\kappa} \Big)\text{”},$$

where this also depends on the connectivity pattern α of the curves as in (4.1), and where

$$I_{D;x_1,\ldots,x_{2N}}(\overline{\eta}) := \Big(\sum_{j=1}^N I_{D;x_{a_j},x_{b_j}}(\eta_j) + 12\, m_D(\overline{\eta}) \Big) - \min_{\overline{\zeta} \in \mathcal{X}_\alpha} \Big(\sum_{j=1}^N I_{D;x_{a_j},x_{b_j}}(\zeta_j) + 12\, m_D(\overline{\zeta}) \Big) \in [0, +\infty]$$

is the associated Loewner energy. See [PW24] for the mathematical formulation.

5.2 Semiclassical limit of the partition/correlation functions

Let us study the $\kappa \to 0$ limit of the pure partition functions (4.4),

$$\mathcal{Z}_\alpha(D; x_1, \ldots, x_{2N}) := \Big(\prod_{j=1}^N H_D(x_{a_j}, x_{b_j}) \Big)^{h_{1,2}} \mathsf{E}_\alpha^\kappa \Big[\mathbb{1}\{\gamma_i \cap \gamma_j = \varnothing \; \forall i \neq j\} \, \exp\Big(\frac{c(\kappa)}{2} \, m_D(\overline{\gamma}) \Big) \Big].$$

Recall that they can be thought of as conformal blocks for degenerate fields of type $\phi_{1,2}$. Since $c(\kappa) \to -\infty$ as $\kappa \to 0$, this is a semiclassical limit for the theory.

Because the quantity $m_D(\overline{\gamma})$ is positive (it cannot equal zero as the curves are macroscopic), we see that for any realization of the random curves $\overline{\gamma}$, the exponential vanishes in the limit:

$$\exp\Big(\frac{c(\kappa)}{2} \, m_D(\overline{\gamma}) \Big) \quad \overset{\kappa \to 0}{\longrightarrow} \quad 0.$$

This does not mean, however, that the limit curves would be non-interacting: as remarked before, the measure of the curves concentrates on “interacting geodesics” in the hyperbolic metric on D.

The limit of the function $\mathcal{Z}_\alpha$ includes interesting geometric information. In fact, we proved in [PW24] that (this can be regarded as a special case of the Zamolodchikov conjecture [Zam86])

$$-\kappa \log \mathcal{Z}_\alpha(D; x_1, \ldots, x_{2N}) \quad \overset{\kappa \to 0}{\longrightarrow} \quad \mathcal{U}_\alpha(D; x_1, \ldots, x_{2N}),$$

where $\mathcal{U}_\alpha$ can be thought of as a semiclassical conformal block and has the formula

$$\mathcal{U}_\alpha(D; x_1, \ldots, x_{2N}) = \min_{\bar{\zeta} \in \mathcal{X}_\alpha} \Big(\sum_{j=1}^N I_{D; x_{a_j}, x_{b_j}}(\zeta_j) + 12\, m_D(\bar{\zeta}) \Big) \; + \; 3 \sum_{j=1}^N H_D(x_{a_j}, x_{b_j}).$$

Moreover, as might not be very surprising, $\mathcal{U}_\alpha$ satisfies the *semiclassical BPZ equations*

$$\frac{1}{2}(\partial_j \mathcal{U}_\alpha(x_1, \ldots, x_{2N}))^2 - \sum_{i \neq j} \frac{2}{x_i - x_j} \partial_i \mathcal{U}_\alpha(x_1, \ldots, x_{2N}) \; = \; \sum_{i \neq j} \frac{6}{(x_i - x_j)^2},$$

for all $j \in \{1, \ldots, 2N\}$. Heuristically, plugging the expression $\mathcal{Z}_\alpha = \exp(-\frac{1}{\kappa}\mathcal{U}_\alpha)$ into the BPZ PDE system (4.6) gives exactly the above equation as $\kappa \to 0$. (The difficulty is to prove that one can exchange the limit and derivative, which we obtain as a byproduct of the LDP result for the SLE curves.) In the physics literature, this appears as a system of Hamilton-Jacobi type equations (also related to Painlevé VI) associated to the semiclassical conformal blocks, see [LLNZ14] and references therein. A semiclassical limit of the dual version of these PDEs (associated to the dual fields of conformal weight $h_{2,1}$ in the Liouville CFT of central charge $c \geq 25$) have been used to give a probabilistic proof for the Takhtajan-Zograf theorem [TZ03] relating Poincaré's accessory parameters to the classical Liouville action [LRV22].

For a relation of the curve endpoints to classical integrable Calogero-Moser type systems, see [ABKM24, AHP24].

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