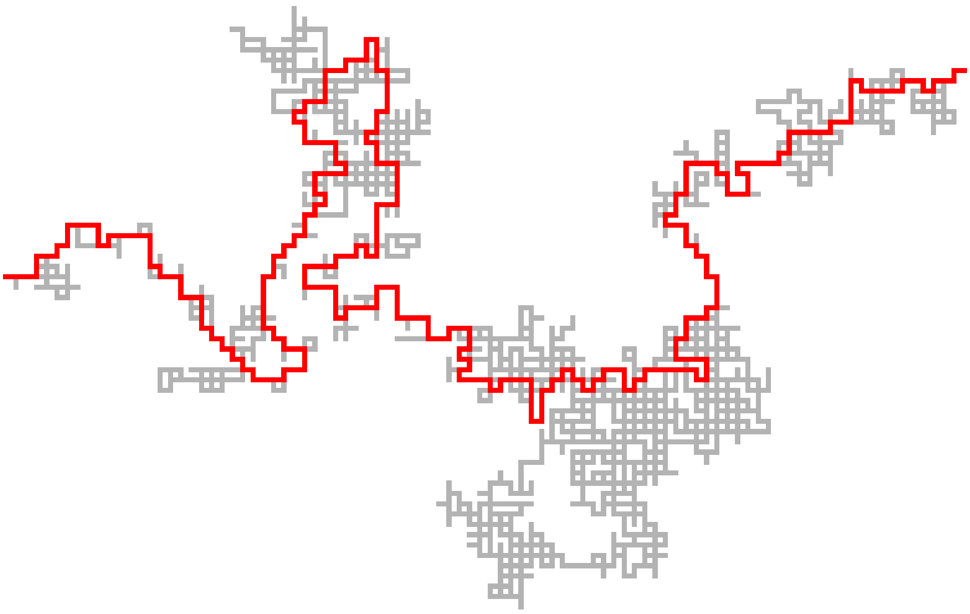
Lectures on
Geometry of random conformally invariant curves

Eveliina Peltola*

Abstract

In planar random geometry, a plethora of conformally invariant and covariant objects has been emerged in the recent years. Among these particularly fruitful have been random fractal curves derived from one-dimensional Brownian motion: Schramm-Loewner evolutions (SLE), conformal loop ensembles (CLE), and their variants. Originally they were introduced in the context of critical models in statistical physics to understand conformal invariance and critical phenomena. Such curves describe scaling limits of lattice interfaces, and level and flow lines of random fields, but have also turned out to be quite interesting by their own right. In these lectures, I will introduce models for conformally invariant random curves and discuss their relation to critical models and some geometric properties.

As general references to random planar curves, see for example Werner's Saint-Flour lecture notes [Wer03], or for more details, the relatively recent book by Kemppainen [Kem17] or the Cambridge lecture notes [BN16] by Berestycki & Norris (especially for probabilists).

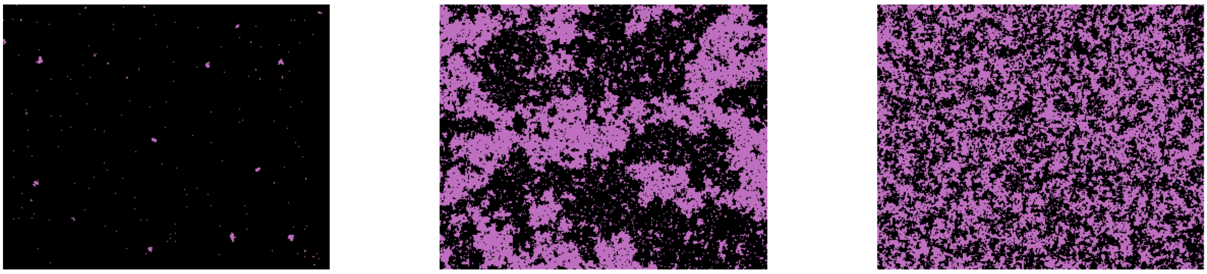


A symmetric random walk on a 200×200 square grid and its loop-erasure (red). The erased loops are gray. (Figure from [KKP19].)

*Aalto University, Department of Mathematics and Systems Analysis. `eveliina.peltola@aalto.fi`
University of Bonn, Institute for Applied Mathematics. `eveliina.peltola@hcm.uni-bonn.de`

These **rough** lecture notes describe the content of the minicourse "Geometry of random conformally invariant curves" at SISSA given in fall 2022. **Please let me know if you find mistakes or misprints!**

Contents



$T < T_c$ $\qquad$ $T = T_c$ $\qquad$ $T_c < T$

Figure 1.1: Illustration of the phase transition of the Ising model. In high temperatures (right), the system is disordered, whereas in low temperatures (left), aligned spins are favored. At criticality, macroscopic clusters of aligned spins appear. The phase transition is continuous in the sense that the magnetization is continuous across the transition at T_c. (Figure from [Pel16].)

1 Introduction: Critical models in statistical physics

- ▷ In statistical physics one studies macroscopic systems consisting of many microscopic random objects. The idea is that the number of objects tends to infinity, which is formulated as a "scaling limit". One wishes in particular to capture *universal* properties that survive in the scaling limit, regardless of the microscopic setup of the model.

- ▷ For example, *Donsker's theorem* states that symmetric random walk converges, when suitably rescaled, to Brownian motion (see [MP10, Chapter 5]). This in fact holds quite universally: regardless of the precise microscopic formulation of the random walk, assuming that the distribution of jumps is nice enough.

- ▷ Donsker's theorem holds for any space dimension. For dimension two, a theorem of Lévy (see [MP10, Theorem 7.20]) states that the limiting object (2D Brownian motion) is *conformally invariant in distribution* (up to a time change) — in particular, its trajectory is a random conformally invariant set in the plane.

Example: Ising model. Conformal invariance is a fundamental property of many models in statistical physics, such as critical Ising model, critical Bernoulli percolation, and other critical models with second order phase transition — see the lecture notes [DCS12] by Duminil-Copin & Smirnov for a detailed account. A prototypical example of a statistical physics model with a continuous (second order) phase transition is the two-dimensional ferromagnetic (nearest-neighbor) *Ising model* (with fixed positive coupling constant). On a finite graph $G = (V, E)$ with vertices V and edges E, a configuration in the Ising model consists of an assignment $\sigma : V \to \{\pm 1\}$ of spins $\sigma_x \in \{\pm 1\}$ to each vertex $x \in V$. The probability of a configuration σ is given by the *Boltzmann distribution* (the canonical ensemble)

$$\mathbb{P}[\sigma] = \frac{e^{-\beta H(\sigma)}}{Z}, \qquad H(\sigma) = -\sum_{(x,y)\in E} \sigma_x \sigma_y, \qquad Z = \sum_{\sigma} e^{-\beta H(\sigma)},$$

where $\beta = \frac{1}{T} > 0$ is the inverse-temperature and $H(\sigma)$ is the Hamiltonian. The Boltzmann distribution favors configurations where the neighboring spins are aligned. The behavior of the system is also highly dependent on the temperature: there is an order-disorder phase transition at a unique critical temperature $\beta_c = \frac{1}{T_c} \in (0, \infty)$. At the critical temperature, the scaling limit of the Ising model is believed (and in may ways proved) to become conformally invariant in the

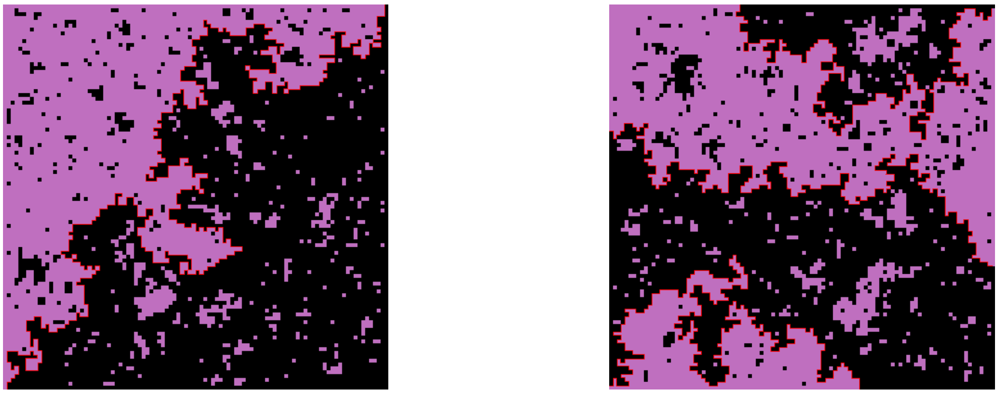


Figure 1.2: Critical Ising model configurations on a square lattice with alternating boundary conditions (that is, some boundary segments have spins equal to +1 and the other segments have spins equal to −1). Interfaces connecting boundary points are highlighted. (Figure from [Pel19].)

scaling limit (e.g., its interfaces and correlation functions converge to conformally invariant or covariant quantities [HS13, CHI15, CDCH+14, Izy17, BPW21]).

To study the geometry of the critical Ising model, one can study *interfaces* between "+" spins and "−" spins. Some of these interfaces are macroscopic, so they survive in the scaling limit. For example, one can force the system to have a macroscopic interface via imposing boundary conditions: take $G = D^\delta = D \cap \delta\mathbb{Z}^2$ for some bounded simply connected domain $D \subsetneq \mathbb{C}$, split the boundary $\partial D = \partial^+ \sqcup \partial^-$ into two segments ∂^+ and ∂^-, and consider the Ising model with the constraint that the vertices in ∂^+ all equal +1 and the vertices in ∂^- all equal −1. (See Figure 1.2.) Then for topological reasons, there must exist a macroscopic path traversing between "+" and "−" spins and connecting the two boundary points where the segments ∂^+ and ∂^- touch. (More generally, one could consider alternating boundary conditions with more "+" and "−" segments on the boundary. In that case, there are several macroscopic boundary-to-boundary interfaces as in Figure 1.2(right).) At the *critical temperature* $T = T_c$, the interfaces have interesting self-similar behavior. Indeed, in the scaling limit $\delta \to 0$, such interfaces have been proven to converge to random conformally invariant curves, called Schramm-Loewner evolution (SLE(3)) curves [CDCH+14, Izy17, BPW21]. Also, loops (domain walls) in the interior of the domain separating "+" spins and "−" spins have been shown to converge to the so-called conformal loop ensemble (CLE(3)) [BH19].

To motivate how one could describe scaling limits of critical Ising interfaces, there are a few natural properties that the limit should satisfy. In addition to *conformal invariance* (predicted via renormalization group methods), one would expect a Markovian property (which holds for many lattice models with local interactions) in the following sense.

Consider the Ising model on G as above, with its boundary divided into the two segments ∂^+ and ∂^-. An exploration process on G, started from one of the boundary points where the segments ∂^+ and ∂^- touch and ending at the other such boundary point, is defined by following the interface between the opposite spins step by step[1]. Let $\gamma(k)$, for $k = 0, 1, \ldots, n$, denote this exploration process (in discrete time). Explore it up to some time k_0. Consider the exploration process $\tilde{\gamma}$ for the model on the smaller grid $\tilde{G} = G \setminus \gamma[0, k_0]$, started from the tip $\gamma(k_0)$, where the boundary conditions are taken as before on ∂G and naturally continued to both sides of the segment $\gamma[0, k_0]$ of $\partial\tilde{G}$. Then, the distribution of the exploration process $\tilde{\gamma}$ associated to the

[1] A careful reader may notice a caveat (that becomes clear by drawing a small figure on the square grid): on the square grid, one might encounter an indetermination for the exploration step, which can be resolved by picking a preferred choice for the direction of each exploration step.

model on the grid $\tilde{G}$ equals the conditional law of the original process γ on the original graph G given the initial segment $\gamma[0, k_0]$. This is called the *(domain) Markov property*.

2 Conformal invariance of 2D Brownian motion

To convey the conceptual ideas of conformal invariance, we will usually consider nice enough planar domains $D \subsetneq \mathbb{C}$. For example, we could assume that the boundary ∂D is a smooth Jordan curve. (Most of the results we will state hold much more generally, but one has to be careful with what happens on the boundary.) A well-known occurrence of conformal invariance for planar objects is the observation that harmonic functions are preserved by conformal maps (Proposition 2.3). From this, one gets to conformal invariance of the Brownian trajectory, or more precisely, the Brownian hull (Proposition 2.6).

2.1 Conformal maps

▷ We say that a map $\varphi : D \to \tilde{D}$ is *conformal* (biholomorphic) if it is holomorphic and bijective. Note that, at any point $z_0 \in D$, a conformal map has a Taylor expansion with the form

$$\varphi(z) = \varphi(z_0) + \varphi'(z_0)\,(z - z_0) + \cdots.$$

Intuitively, one can think of the first term as a *translation* by $\varphi(z_0)$, and the second term as a *rotation* by the argument of $\varphi'(z_0)$ together with a *scaling* by the modulus $|\varphi'(z_0)|$. Thus, up to terms of order $|z - z_0|^2$, the conformal map φ is a composition of a translation, a rotation, and a scaling. One sometimes says that a conformal map is locally angle-preserving.

▷ Recall also that a holomorphic map φ satisfies the *Cauchy-Riemann equations*: writing $z = x + \mathfrak{i}y$ and $\varphi(z) = u(z) + \mathfrak{i}v(z)$, we have $\partial_x u = \partial_y v$ and $\partial_x v = -\partial_y u$.

▷ We denote by $\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}$ the open unit disc in the complex plane $\mathbb{C}$. It is a standard example of a *simply connected domain*, that is, a non-empty open connected planar set (domain) which has no holes (i.e., having trivial fundamental group). In fact, all simply connected domains are conformally equivalent to either $\mathbb{D}$ (see Theorem 2.1), or $\mathbb{C}$. (More generally, among simply connected Riemann surfaces we should also include the Riemann sphere $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$.)

Theorem 2.1 (Riemann mapping theorem). *Let $D \subsetneq \mathbb{C}$ be a simply connected domain. Then, there exists a conformal map $\varphi : D \to \mathbb{D}$. This map is uniquely determined by fixing three degrees of freedom, for instance:*

▷ *picking some point $z_0 \in D$ and requiring that $\varphi(z_0) = 0$ and $\varphi'(z_0) > 0$; or*

▷ *picking three points $a, b, c \in \partial D$ and three points $a', b', c' \in \partial D$, both in counterclockwise order, and requiring that $\varphi(a) = a'$, $\varphi(b) = b'$, and $\varphi(c) = c'$. (This is possible when the boundary ∂D is nice enough so that the map φ is defined at the points $a, b, c \in \partial D$.)*

Proof idea. This is a standard result in complex analysis. There are many proofs for the existence, see for example [Ahl79]. The uniqueness follows by classifying all conformal self-maps of the unit disc $\mathbb{D}$, which is a three-dimensional real Lie group. □

2.2 Dirichlet problem and conformal invariance

▷ Recall that the *Laplacian operator* in $\mathbb{R}^n$ is defined as $\Delta = \partial_1^2 + \partial_2^2 + \cdots + \partial_n^2$. When $n = 2$, we identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$, writing $z = x + \mathfrak{i}y \in \mathbb{C}$ with $(x, y) \in \mathbb{R}^2$, so that $\Delta = \partial_x^2 + \partial_y^2$.

▷ We say that a map $f : D \to \mathbb{R}$ is *harmonic* in $D \subset \mathbb{C}$ if $\Delta f(z) = 0$ for all $z \in D$.

▷ Real and imaginary parts of holomorphic maps give examples of harmonic functions.

Theorem 2.2 (Dirichlet problem)**.** *Let D be a bounded simply connected domain whose boundary ∂D is a smooth Jordan curve. Let $u : \partial D \to \mathbb{R}$ be a continuous function. Then, there exists a unique function $f : \overline{D} \to \mathbb{R}$ which is continuous in $\overline{D}$, harmonic in D, and has u as its boundary values: $f(x) = u(x)$ for all $x \in \partial D$.*

Proof idea. This is a well-known fact in potential theory, that can be found in many books. For example, see [MP10, Theorem 3.12] for a statement involving Brownian motion. □

Remarkably, harmonic functions in the plane are conformally invariant in the following sense.

Proposition 2.3 (Conformal invariance of Dirichlet problem)**.** *Let $\varphi : \tilde{D} \to D$ be a conformal map and f the harmonic function on D with boundary values u. Then $f \circ \varphi$ is the harmonic function on $\tilde{D}$ with boundary values $u \circ \varphi$.*

Proof idea. Given f, one can calculate $f \circ \varphi$ and use the Cauchy-Riemann equations that the conformal map φ has to satisfy to deduce that $\Delta f = 0$ implies $\Delta(f \circ \varphi) = 0$. □

2.3 2D Brownian motion and conformal invariance

There are numerous books concerning Brownian motion. For an extensive reference, see [MP10]. Recall that one-dimensional *Brownian motion* started at a point $B_0 = x \in \mathbb{R}$ is a continuous-time real-valued stochastic process $B = (B_t)_{t \geq 0}$ satisfying the following properties (that determine it uniquely):

▷ (Independent increments): For any partition $0 \leq t_0 < t_1 < t_2 < \cdots < t_n$, the increments $\{B_{t_{j+1}} - B_{t_j} \mid j = 0, 1, \ldots, n-1\}$ are independent random variables.

▷ (Stationary, Gaussian increments): For each $0 \leq s < t$, the increment $B_t - B_s$ has the Gaussian distribution: $B_t - B_s \sim N(0, t-s)$, that only depends on the time difference.

▷ (Continuous sample paths): The map $t \mapsto B_t$ is continuous almost surely.

Definition 2.4. *Two-dimensional (2D) Brownian motion* started at a point $B_0 = z = x + \mathfrak{i}y \in \mathbb{C}$ is a continuous-time real-valued stochastic process $B = (B_t)_{t \geq 0}$ whose real and imaginary parts $(B^{(1)}, B^{(2)})$ are *independent* one-dimensional Brownian motions started at $(B_0^{(1)}, B_0^{(2)}) = (x, y)$:

$$B_t = B_t^{(1)} + \mathfrak{i} B_t^{(2)}, \quad t \geq 0.$$

We denote by $\mathbb{E}_z$ the expected value with respect to the law $\mathbb{P}_z$ of 2D Brownian motion started at z.

The Laplacian operator Δ is the generator of Brownian motion as a Markov process. In particular, it is natural to expect that the conformal invariance of Dirichlet problem yields conformal invariance for 2D Brownian motion. This is indeed the case, but one has to make a time change to account for the different speeds of the Brownian motion in different domains. To prove this, one can use stochastic analysis and the Cauchy-Riemann equations — for details, see [Kem17, Chapter 2.5] or [BN16, Chapter 2.1], or [MP10, Theorem 7.20]. We will instead consider the trajectory of the 2D Brownian path, which is a random planar set (see Proposition 2.6).

In fact, the Dirichlet problem can be solved using 2D Brownian motion.

Proposition 2.5 (Kakutani's formula)**.** *Let D be a bounded simply connected domain whose boundary ∂D is a smooth Jordan curve. Let $u : \partial D \to \mathbb{R}$ be a continuous function. Consider 2D Brownian motion B on D started at $z \in D$ up to the exit time*

$$\tau_D := \inf\{t \geq 0 \mid B_t \notin D\} = \inf\{t \geq 0 \mid B_t \in \partial D\}.$$

The function $z \mapsto \mathbb{E}_z[u(B_{\tau_D})]$ solves the Dirichlet problem in D with boundary values u.

Proof idea. This can be proven, e.g., by using Itô calculus — see [BN16, Chapter 2] for a detailed proof (and [MP10, Theorem 3.12] for a quite general statement). □

One way to determine the law of a random compact subset $K \subset \overline{D}$ (that is, to define a probability distribution for it), is to let the law of K be determined by the collection of probabilities that K doesn't intersect an obstacle A:

$$\big\{\mathbb{P}[K \cap A = \varnothing] \mid A \subset \overline{D} \text{ compact such that } D \smallsetminus A \text{ is simply connected}\big\}.$$

One can check that the sigma-field generated by the events $\{K \cap A = \varnothing\}$ as above coincides with the Borel sigma-field induced by the Hausdorff metric on the space of closed subsets of $\overline{D}$. See the lecture notes [Wer05] for more discussion on this approach.

Intuitively, the *filling* (or *hull*) $\mathrm{fill}_D(B) = \mathrm{fill}(B_{[0,\tau_D]})$ of 2D Brownian motion on D is the random fractal set obtained by looking at the trajectory of the Brownian motion from outside, that is, by filling all the loops in the bounded components of the complement of $B_{[0,\tau_D]}$ in $\mathbb{C}$. The filling is conformally invariant in distribution in the following sense.

Proposition 2.6 (Conformal invariance of Brownian hull)**.** *Let $\varphi : \tilde{D} \to D$ be a conformal map and $\tilde{B}$ and B respectively Brownian motions on $\tilde{D}$ and D started at $z \in \tilde{D}$ and $\varphi(z) \in D$. Then*

$$\mathrm{fill}_D(B) \qquad \text{and} \qquad \mathrm{fill}_D(\varphi \circ \tilde{B})$$

have the same law as random subsets of $\overline{D}$ (as above).

Proof idea. The idea to prove this is related to "conformal restriction" [LSW03, Vir03]. Note that for any obstacle A, we have

$$\begin{aligned}\mathbb{P}[\mathrm{fill}_D(B) \cap A = \varnothing] &= \mathbb{P}_{\varphi(z)}[B \text{ exits } D \text{ before it hits } A] \\ &= \mathbb{P}_{\varphi(z)}[B \text{ exits } D \smallsetminus A \text{ through } \partial D \smallsetminus \partial A] \\ &= \mathbb{E}_{\varphi(z)}[\mathbb{1}\{B_{\tau_{D \smallsetminus A}} \in \partial D \smallsetminus \partial A\}].\end{aligned}$$

Define

$$f : w \mapsto \mathbb{E}_w[\mathbb{1}\{B_{\tau_{D \smallsetminus A}} \in \partial D \smallsetminus \partial A\}], \qquad w \in D \smallsetminus A,$$

which is, by Kakutani's formula (Proposition 2.5), the *harmonic measure* on $D \smallsetminus A$ of the set $\partial D \smallsetminus \partial A$, that is, the unique harmonic function in $D \smallsetminus A$ that has boundary values 1 at $\partial D \smallsetminus \partial A$, and 0 at $D \cap \partial A$. We know from the conformal invariance of Dirichlet problem that this function is conformally invariant. More precisely, define $\tilde{A} = \varphi^{-1}(A)$ and

$$\tilde{f} : z \mapsto \mathbb{E}_z[\mathbb{1}\{\tilde{B}_{\tau_{\tilde{D} \smallsetminus \tilde{A}}} \in \partial \tilde{D} \smallsetminus \partial \tilde{A}\}], \qquad z \in \tilde{D} \smallsetminus \tilde{A},$$

which is the harmonic measure on $\tilde{D} \smallsetminus \tilde{A}$ of the set $\partial \tilde{D} \smallsetminus \partial \tilde{A}$. Then, applying Proposition 2.3 to the conformal map φ restricted to the set $\tilde{D} \smallsetminus \tilde{A}$, we see that $f \circ \varphi = \tilde{f}$, which gives the assertion:

$$\mathbb{P}[\mathrm{fill}_D(B) \cap A = \varnothing] = \mathbb{E}_{\varphi(z)}[\mathbb{1}\{B_{\tau_{D \smallsetminus A}} \in \partial D \smallsetminus \partial A\}]$$

$$
\begin{aligned}
&= f(\varphi(z)) = \tilde{f}(z) \\
&= \mathbb{E}_z[\mathbb{1}\{\tilde{B}_{\tau_{\tilde{D}\setminus\tilde{A}}} \in \partial\tilde{D} \setminus \partial\tilde{A}\}] \\
&= \mathbb{P}_z[\tilde{B} \text{ exits } \tilde{D} \setminus \tilde{A} \text{ through } \partial\tilde{D} \setminus \partial\tilde{A}] \\
&= \mathbb{P}_z[\tilde{B} \text{ exits } \tilde{D} \text{ before it hits } \tilde{A}] \\
&= \mathbb{P}[\mathrm{fill}_D(\varphi \circ \tilde{B}) \cap A = \varnothing].
\end{aligned}
$$

□

Remark 2.7. $\mathrm{fill}_D(B)$ is a random fractal set, whose outer boundary has Hausdorff dimension 4/3 (this property, known as Mandelbrot conjecture, was proven using by SLE in [LSW01]).

3 Conformal invariance emerging for discrete models

It is quite difficult to study the geometry of random planar sets that have double points. Even worse, the trajectory of Brownian motion has points of arbitrary multiplicity. Hence, it would be meaningful to first study *simple* (that is, injective) planar curve models that exhibit conformal invariance. We will next consider simple curves obtained from random walk, whose scaling limits should be simple conformally invariant curves.

3.1 Simple curves from random walk

Consider the symmetric random walk X on $\mathbb{Z}^2$, say. To get a simple path from X, one can:

- ▷ either make X simple by erasing all loops from it (LERW, see Section 3),
- ▷ or condition X to be a simple (self-avoiding) path (SAW, see Section 5).

The SAW model in the plane has turned out to be extremely difficult to study rigorously, being very far from a Markov process. The LERW model was originally introduced as a toy model for SAW (by Greg Lawler in his PhD thesis; see [Law16] for relatively recent notes), but in two and three dimensions, it actually belongs to a different universality class (that is, its scaling limit has different macroscopic properties, such as Hausdorff dimension) than SAW. In dimensions five and higher, these models coincide in the scaling limit with that of the symmetric random walk (Brownian motion), roughly because there is so much space that the self-avoidance constraint becomes redundant in sufficiently high spatial dimension[2]. Let us now focus on the LERW model in the plane.

- ▷ A *walk* on $\mathbb{Z}^2$ started at a given point $x_0 \in \mathbb{Z}^2$ is a nearest-neighbor path $\underline{x} = (x_0, x_1, \ldots, x_m)$ of vertices of $\mathbb{Z}^2$, or equivalently, comprising edges $(\{x_0, x_1\}, \{x_1, x_2\}, \ldots, \{x_{m-1}, x_m\})$ of $\mathbb{Z}^2$.
- ▷ The *loop-erasure* $\mathrm{LE}(\underline{x})$ of a walk $\underline{x}$ is obtained from it by chronologically erasing all loops:

  $$\mathrm{LE}(\underline{x})_0 := x_0, \quad \mathrm{LE}(\underline{x})_{k+1} := x_{n_k+1}, \; k \geq 0,$$

  where $n_k := \max\{n \leq m \mid x_n = \mathrm{LE}(\underline{x})_k\}$. Note that the number of steps in $\mathrm{LE}(\underline{x})$ can vary.
- ▷ If X is a symmetric random walk X on $\mathbb{Z}^2$, then its loop-erasure $L := \mathrm{LE}(X)$ is a random walk on $\mathbb{Z}^2$, called loop-erased random walk (LERW).

[2]It is also believed that in dimension four, both LERW and SAW converge to Brownian motion, but only the case of LERW has been proven rigorously (by Lawler). In dimension three, very little is known of SAW, while it is known that LERW has a scaling limit [Koz07], that has still not been characterized very precisely.

We consider LERWs of finite length living in a (simply connected) discrete domain $G = D \cap \mathbb{Z}^2$ started at some fixed point $x_0 = X_0 = L_0$. Denote the exit time of the walk from G as

$$\tau = \tau_G := \inf\{m \in \mathbb{N} \mid X_m \notin G\}.$$

The loop-erasure up to time τ is the simple discrete path $L = \mathrm{LE}((X_k)_{0 \le k \le \tau})$. We write σ for the length of L, so that $L_\sigma = X_\tau \in \partial G$. Note that σ is a random stopping time (for the natural filtration generated by the walk). The distribution of the exit point X_τ on ∂G is the same as that for the LERW L_σ. This is just the discrete harmonic measure, that converges (under suitable regularity conditions) to the harmonic measure, i.e., the distribution of the exit point of Brownian motion from D. This, in turn, is closely related to Dirichlet problem and thus it would be natural to expect that, analogously to the Brownian hull, the scaling limit of the LERW path would be conformally invariant. This is indeed the case (see Theorem 4.9).

3.2 Markov property for LERW

In addition to conformal invariance, another natural property is helpful in determining the scaling limit of LERW — the (domain) *Markov property*. This property is apparent for exploration processes of interfaces in critical statistical physics models, such as the critical Ising model and critical percolation, while for the LERW it's not completely obvious (see Proposition 3.1).

It is convenient to condition the walk upon exiting at a given $y_0 = L_\sigma = X_\tau \in \partial G$, which we will assume throughout. We say that the LERW *in G is started from x_0 and ending at y_0.*

Proposition 3.1 (Markov property). *Fix $y_0 \in \partial G$ and $y_1, \ldots, y_k \in G$ which are possible last steps of $L = (L_0, L_1, \ldots, L_\sigma)$. Then, the conditional law of $(L_0, L_1, \ldots, L_{\sigma-k-1})$ on the event*

$$\{L_\sigma = y_0, L_{\sigma-1} = y_1, \ldots, L_{\sigma-k} = y_k\}$$

is LERW in the discrete slit domain $G \smallsetminus \{y_1, \ldots, y_k\}$ started from x_0 and ending at y_k.

Proof idea. For a vertex z and a set A, the Green's function $\mathcal{G}(z, A) = \mathcal{G}_{x_0}(z, A)$ is defined as the expected number of visits of random walk to z (started at x_0) before hitting A:

$$\mathcal{G}(z, A) = \sum_{j=0}^{\iota_A - 1} \mathbb{P}[X_j = z],$$

where $\iota_A := \inf\{j \in \mathbb{N} \mid X_j \in A\}$ is the hitting time of A. One can check that, for vertices $w_0 = x_0$, $w_1, w_2, \ldots, w_{\sigma-1}$, and $w_\sigma = y_0$, the following identity holds:

$$\begin{aligned}
&\mathbb{P}\left[L = (w_0, w_1, \ldots, w_{\sigma-1}, w_\sigma)\right] \\
&= \sum_{\substack{\underline{x} = (x_0, x_1, \ldots, x_m) : \\ \mathrm{LE}(\underline{x}) = (w_0, w_1, \ldots, w_\sigma)}} \mathbb{P}[X = (x_0, x_1, \ldots, x_m)] \\
&= \mathcal{G}(w_0, \partial G) \times \mathbb{P}[\text{step } w_0 \to w_1] \times \mathcal{G}(w_1, \partial G \cup \{w_0\}) \times \mathbb{P}[\text{step } w_1 \to w_2] \times \cdots \times \\
&\quad \times \cdots \times \mathcal{G}(w_{\sigma-1}, \partial G \cup \{w_0, w_1, \ldots, w_{\sigma-2}\}) \times \mathbb{P}[\text{step } w_{\sigma-1} \to w_\sigma].
\end{aligned}$$

Next, note that the product of Green functions

$$\prod_{j=0}^{\sigma-1} \mathcal{G}(w_j, \partial G \cup \{w_0, w_1, \ldots, w_{j-1}\})$$

is symmetric under permutations of the points $(w_0, w_1, \ldots, w_{\sigma-1})$. This follows by the property

$$\mathcal{G}(z, A)\, \mathcal{G}(z', A \cup \{z\}) = \mathcal{G}(z', A)\, \mathcal{G}(z, A \cup \{z'\}),$$

Figure 3.1: Comparison of the loop-erasures when a small loop (of size ε) is erased (left) and when the small loop is not erased (right). The resulting loop-erasures (highlighted red) are quite different.

that is, making loops at z and then making loops at z' without going back to z is the same as making loops at z' and then making loops at z without going back to z'. This gives the claim with $k = 1$: given y_0 and y_1, the conditional law is

$$\mathbb{P}[L = (w_0, w_1, \ldots, w_{\sigma-1}, w_\sigma) \mid L_\sigma = y_0, L_{\sigma-1} = y_1]$$
$$= \frac{\mathcal{G}(y_1, \partial G) \times \mathbb{P}[\text{step } y_1 \to y_0]}{\mathbb{P}[L_\sigma = y_0, L_{\sigma-1} = y_1]} \times \underbrace{\prod_{j=0}^{\sigma-2} \Big(\mathcal{G}(w_j, (\partial G \cup \{y_1\}) \cup \{w_0, w_1, \ldots, w_{j-1}\}) \times \mathbb{P}[\text{step } w_j \to w_{j+1}] \Big)}_{\text{LERW in slit domain } G \smallsetminus \{y_1\}},$$

where $w_{\sigma-1} = y_1$, and where $\partial G \cup \{y_1\}$ is the boundary of the slit domain $G \smallsetminus \{y_1\}$ with one point y_1 removed. This gives the claim for $k = 1$, and one can iterate to get the claim for general $k \geq 2$. The details are left as an exercise. □

3.3 Scaling limit of LERW: heuristics

Consider LERW γ^δ in $D^\delta = D \cap \delta\mathbb{Z}^2$ started from x_0^δ and ending at $y_0^\delta \in \partial D^\delta$, where $x_0^\delta \to x_0 \in D$ and $y_0^\delta \to y_0 \in \partial D$ as $\delta \to 0$. We wish to describe the limit of γ^δ as $\delta \to 0$. From Donsker's theorem we know that the underlying random walk converges to 2D Brownian motion. Hence, we would expect that LERW converges to the loop-erasure of Brownian motion. However, there is no obvious loop-erasure procedure for Brownian motion[3]:

- ▷ The probability that Brownian motion visits any given point is zero.
- ▷ Brownian motion has points of arbitrary multiplicity.
- ▷ There is no well-defined "first" loop, and loops occur at all scales.
- ▷ Trying to re-normalize changes the geometry of the obtained loop-erasure: if we wish to erase loops of size larger than some given ε, then the loop-erasure *can depend drastically on* ε. See Figure 3.1.

Anyway, since the loop-erasure is a purely geometric property, we would expect that if LERW has a scaling limit, it satisfies conformal invariance. (One can also argue that the limit object must be a simple curve by ruling out "almost" loops, see [Sch00].)

Conformal invariance. Since we expect conformal invariance, it should be sufficient to look at the model in the discretization $D^\delta = \mathbb{D} \cap \delta\mathbb{Z}^2$ of the unit disc $\mathbb{D}$. The LERW lives in D^δ and goes from 0 to 1, and we can view it backwards as a simple curve from 1 to 0 as in the Markov property (Proposition 3.1). We thus consider simple curves $\gamma = \gamma^{\mathbb{D};1,0} : [0,\infty) \to \overline{\mathbb{D}}$ (with some parameterization to be determined) such that $\gamma(0) = 1 \in \partial\mathbb{D}$ and $\lim_{t\to\infty} \gamma(t) =: \gamma(\infty) = 0$.

For each time t, the Riemann mapping theorem (Theorem 2.1) shows that there exists a unique conformal map $f_t : \mathbb{D} \smallsetminus \gamma[0,t] \to \mathbb{D}$ such that $f_t(0) = 0$ and $f_t(\gamma(t)) = 1$.

[3]There is a paper [Zha11] (that appeared later than [LSW04]) showing that one can indeed define a loop-erasure, using a coupling technique, and such a loop-erasure applied to 2D Brownian motion indeed yields SLE(2).

Lemma 3.2. *The map $t \mapsto |f_t'(0)|$ is increasing and continuous on $[0,\infty)$, and satisfies*

$$\lim_{t\to\infty} |f_t'(0)| = \infty.$$

Proof idea. This can be proven using Schwarz lemma, which is a basic and very useful tool in complex analysis (see, e.g. [BN16, Lemma 1.2]). □

Since $f_0 = \mathrm{id}_{\mathbb{D}}$, from Lemma 3.2 we see that we can parameterize time as $t = \log|f_t'(0)|$, i.e.,

$$|f_t'(0)| = e^t.$$

Hence, to each realization of the curve $\gamma : [0,\infty) \to \overline{\mathbb{D}}$, we can associate a unique collection $(f_t)_{t\geq 0}$ of conformal maps chosen as above.

Conformal invariance of the law of the curve γ states that we can determine the law in any other simply connected domain D via a conformal map.

Definition 3.3 (Conformal invariance)**.** For a random curve $\gamma^{D;y_0,x_0}$ in $\overline{D}$ from $y_0 \in \partial D$ to $x_0 \in D$, the law $\mathbb{P}^{D;y_0,x_0}$ of $\gamma^{D;y_0,x_0}$ is given by the pushforward $\varphi_* \mathbb{P}^{\mathbb{D};1,0}$ of the law $\mathbb{P}^{\mathbb{D};1,0}$ of $\gamma^{\mathbb{D};1,0}$ by the conformal map $\varphi : \mathbb{D} \to D$ sending 1 to $y_0 = \varphi(1)$ and 0 to $x_0 = \varphi(0)$.

Markov property. Given a segment $\gamma[0,t]$, we would like to say that the conditional law of the rest $\gamma[t,\infty)$ of the curve is similar to the original model but in the slit domain $\mathbb{D} \smallsetminus \gamma[0,t]$.

- ▷ On the one hand, the Markov property of LERW (Proposition 3.1) suggests that the law of $\gamma[t,\infty)$ should be the limit as $\delta \to 0$ of the (backward) LERW in $D^\delta \smallsetminus \gamma^\delta[0,t]$ from $\gamma^\delta(t)$ to 0.
- ▷ On the other hand, the conformal invariance would imply that the models in $\mathbb{D}$ and $\mathbb{D}\smallsetminus\gamma[0,t]$ are related by the conformal map f_t: the law of $\gamma[t,\infty)$ should be the same as the law of the image $f_t^{-1}(\tilde{\gamma}[0,\infty))$, where $\tilde{\gamma} \sim \gamma^{\mathbb{D};1,0}$ is an independent copy of γ in $\overline{\mathbb{D}}$ from 1 to 0.

Hence, the curve γ should satisfy the following *conformal Markov property*:

Definition 3.4. The conditional law of $\gamma[t,\infty)$ given $\gamma[0,t]$ under $\mathbb{P}^{\mathbb{D};1,0}$ is

- ▷ (Domain Markov property) the same as $\mathbb{P}^{\mathbb{D}\smallsetminus\gamma[0,t];\gamma(t),0}$,
- ▷ (Conformal Markov property) which is also the same as the pushforward law $(f_t^{-1})_* \mathbb{P}^{\mathbb{D};1,0}$.

Remark 3.5. In fact, one can think of generating γ by iteration of conformal maps. One can check that $(f_{t+s})_{s\geq 0}$ and $(\tilde{f}_s \circ f_t)_{s\geq 0}$ have the same law (here $\tilde{f}$ is an independent copy of f), since the derivatives of the maps at the origin just multiply upon composition (because the maps preserve the origin). Hence, we morally have that, for fixed $n \in \mathbb{N}$ and $t > 0$, the map f_t is a composition of copies of $\tilde{f}_{t/n}$. One can think of these as *independent and stationary increments.*

4 Loewner's theorems and SLE(κ)

4.1 From curves to conformal maps

In the 1920s, Charles Loewner observed that any simple planar curve $\gamma : [0,\infty) \to \overline{\mathbb{D}}$ such that $\gamma(0) = 1 \in \partial\mathbb{D}$ and $\lim_{t\to\infty} \gamma(t) =: \gamma(\infty) = 0$ can be encoded in a one-dimensional function $\zeta : [0,\infty) \to \mathbb{S}^1 = \partial\mathbb{D}$, called a *driving function.* This is, in a sense, related to the winding of the curve — or rather the conformal map f_t. Namely, we can write

$$\zeta_t = \frac{|f_t'(0)|}{f_t'(0)}.$$

Put differently, let $g_t : \mathbb{D} \smallsetminus \gamma[0,t] \to \mathbb{D}$ be the unique conformal map such that $g_t(0) = 0$ and $g_t'(0) = e^t$. Then

$$\zeta_t = g_t(\gamma(t)).$$

Theorem 4.1 (Loewner chain from simple curve)**.** *The maps $(g_t)_{t\geq 0}$ satisfy the initial value problem*

$$\partial_t g_t(z) = -g_t(z)\,\frac{g_t(z) + \zeta_t}{g_t(z) - \zeta_t}, \qquad g_0(z) = z \tag{LE}$$

for all $z \in \overline{\mathbb{D}}$, where the solution $(g_t(z))_{t\in[0,T_z)}$ is defined up to the blow-up time (called swallowing time of the point z)

$$T_z := \sup\{t \geq 0 \mid \liminf_{s\nearrow t} |g_s(z) - \zeta_s| > 0\} \;\in\; [0, +\infty]. \tag{4.1}$$

In other words, the conformal maps g_t are defined in $\mathbb{D}$ off the curve $\gamma[0,t]$. (One can also consider a curve that has self-touchings, in which case the maps g_t are defined in the connected component of $\mathbb{D} \smallsetminus \gamma[0,t]$ containing the origin. Even more generally, one could consider "locally growing" sets, see [Kem17, Chapter 4] for details.) The maps $(g_t)_{t\geq 0}$ are often referred to as a *Loewner chain*, and the differential equation (LE) is called *Loewner equation*.

Proof idea. Consider first what happens at an infinitesimal time increment: one can show that

$$\partial_t g_t(z)\Big|_{t=0} = \lim_{t\searrow 0} \frac{g_t(z) - g_0(z)}{t} = \lim_{t\searrow 0} \frac{g_t(z) - z}{t} = -z\frac{z+1}{z-1}.$$

The heuristic idea is that, when $t > 0$ is very small, the curve segment $\gamma[0,t]$ is very small, and the conformal map g_t collapsing $\gamma[0,t]$ to the boundary and fixing the origin is related to a vector field on the disc $\mathbb{D}$ that is

- ▷ zero at the origin (since the origin is fixed),
- ▷ has a pole at the boundary point 1 (since the tip $\gamma(t)$ is close to 1),
- ▷ zero at the boundary point -1 (by symmetry),
- ▷ and is tangential to the boundary $\partial\mathbb{D}$.

Making this rigorous is non-trivial — for the chordal case, a detailed proof is given in [Kem17, Chapter 4.2.1]. (The radial case follows from the chordal case, and the latter is slightly easier.)

Note the following composition property for the conformal maps.

Lemma 4.2. *The maps $(g_t)_{t\geq 0}$ have the following composition rule: writing for each fixed $t \geq 0$*

$$\tilde{\gamma}^{(t)}(\varepsilon) = \overline{g_t(\gamma[t, t+\varepsilon])}, \quad \varepsilon \geq 0,$$

if $(\tilde{g}^{(t)}_\varepsilon)_{\varepsilon\geq 0}$ are the maps associated to $\tilde{\gamma}$, then

$$g_{t+\varepsilon} = \tilde{g}^{(t)}_\varepsilon \circ g_t.$$

Proof idea. This follows from a computation: since the maps g_t preserve the origin, the derivatives of the maps at the origin just multiply upon composition, so the time-parameterization by $g_t'(0) = e^t$ is preserved. □

To derive the asserted differential equation (LE), we can use the above lemma and the chain rule. Indeed, for fixed t, using the chain rule and applying the above observation to $\tilde{g}$ gives

$$\begin{aligned}\partial_\varepsilon (g_{t+\varepsilon} \circ g_t^{-1})(z)\Big|_{\varepsilon=0} &= \partial_\varepsilon \tilde{g}_\varepsilon^{(t)}(z)\Big|_{\varepsilon=0} \\ &= \lim_{\varepsilon \searrow 0} \frac{\tilde{g}_\varepsilon^{(t)}(z) - \tilde{g}_0^{(t)}(z)}{\varepsilon} = \lim_{\varepsilon \searrow 0} \frac{\tilde{g}_\varepsilon^{(t)}(z) - z}{\varepsilon} = -z \frac{z + \zeta_t}{z - \zeta_t},\end{aligned}$$

since when $\varepsilon > 0$ is very small, the conformal map $\tilde{g}_\varepsilon^{(t)}$ collapsing $\tilde{\gamma}^{(t)}[0, \varepsilon]$ to the boundary and fixing the origin is related to a vector field on the disc $\mathbb{D}$ that is

- ▷ zero at the origin (since the origin is fixed),
- ▷ has a pole at the boundary point $\zeta_t = g_t(\gamma(t))$ (since the tip $\tilde{\gamma}^{(t)}(\varepsilon)$ is close to ζ_t),
- ▷ zero at the boundary point $-\zeta_t$ (by symmetry),
- ▷ and is tangential to the boundary $\partial\mathbb{D}$.

Finally, since

$$g_{t+\varepsilon} = g_{t+\varepsilon} \circ g_t^{-1} \circ g_t,$$

using the chain rule again gives

$$\partial_t g_t(z) = \lim_{\varepsilon \to 0} \frac{g_{t+\varepsilon}(z) - g_t(z)}{\varepsilon} = -g_t(z) \frac{g_t(z) + \zeta_t}{g_t(z) - \zeta_t},$$

where $\zeta_t = g_t(\gamma(t))$. □

4.2 From conformal maps to growing sets

Conversely, one can show that given a continuous function $\zeta : [0, \infty) \to \mathbb{S}^1$, Loewner's initial value problem (LE) for fixed $z \in \overline{\mathbb{D}}$ has a unique solution, which is a conformal map from a subset of $\mathbb{D}$ onto $\mathbb{D}$, defined off of some compact set $K_t \subset$ called a hull.

Theorem 4.3 (Loewner chain from driving function)**.** *Let $\zeta : [0, \infty) \to \mathbb{S}^1$ be a continuous function such that $\zeta_0 = 1$. Then for each $z \in \overline{\mathbb{D}}$, the initial value problem* (LE) *has a unique solution $(g_t(z))_{t \in [0, T_z)}$ defined up to the blow-up time* (4.1)*. Moreover, for each $t \geq 0$, the map $z \mapsto g_t(z)$ is a conformal map from $\mathbb{D} \smallsetminus K_t$ onto $\mathbb{D}$, where*

$$K_t = \{z \in \overline{\mathbb{D}} \mid T_z \leq t\}.$$

Proof idea. The existence and uniqueness of the solution follows from standard ODE theory (Picard-Lindelöf, or Cauchy-Lipschitz theorem). To show that $z \mapsto g_t(z)$ is a conformal map, we need to argue that:

1. $z \mapsto g_t(z)$ is holomorphic on $\mathbb{D} \smallsetminus K_t$,
2. $z \mapsto g_t(z)$ is injective on $\mathbb{D} \smallsetminus K_t$,
3. and $g_t(\mathbb{D} \smallsetminus K_t) = \mathbb{D}$.

From the differential equation (LE) one can derive a differential equation

$$\partial_t g_t'(z) = -g_t'(z) \frac{(g_t(z))^2 - 2\,\zeta_t\, g_t(z) - \zeta_t^2}{(g_t(z) - \zeta_t)^2},$$

which gives the existence of the complex derivative $g_t'(z)$ and shows holomorphicity (Property 1). Similarly one finds a differential equation for $g_t(z) - g_t(w)$, which shows that $g_t(z) \neq g_t(w)$ when $z \neq w$, which yields injectivity (Property 2). The surjectivity (Property 3) is obtained by considering the Loewner flow backwards in time, which allows to solve for the equation $g_t(z) = w$ with given $w \in \mathbb{D}$: for times $s \in [0,t]$, the backwards flow is $h_s : \mathbb{D} \to h_s(\mathbb{D})$

$$\partial_s h_s(w) = -h_s(w)\,\frac{h_s(w) + \zeta_{t-s}}{h_s(w) - \zeta_{t-s}}, \qquad h_0(w) = w,$$

and we have

$$g_t(h_t(w)) = w \quad \Longrightarrow \quad z = h_t(w).$$

For the chordal case, a detailed proof is given in [Kem17, Chapter 4.2.2]. □

Remark 4.4. We don't know how the hulls K_t look like in general. It is very non-trivial to try to show that K_t would be generated by a curve. This can be done in some situations, e.g., when ζ is 1/2-Hölder with Hölder norm strictly less than 4, see [MR05]. When ζ is given by a one-dimensional Brownian motion, one can also prove that the obtained hulls are generated by a continuous curve, but this is very technical, see [RS05].

4.3 Scaling limit of LERW: radial SLE

The upshot from Loewner's theorems is that we can try to describe the sought scaling limit curve γ in terms of a one-dimensional process $\zeta_t = \exp(\mathfrak{i}W_t)$, where $W : [0,\infty) \to \mathbb{R}$ is chosen suitably. It turns out that for the scaling limit of LERW (and any other model whose scaling limit should satisfy *conformal invariance* and *domain Markov property*), W has to be a multiple of Brownian motion (plus possibly a drift). Indeed, we expect that the following properties hold:

1. $t \mapsto W_t$ is continuous,
2. the increments $W_{t+s} - W_t$ are independent (by the iteration idea and domain Markov property) and stationary (only depend on s),
3. and there is an additional symmetry in law $W \leftrightarrow -W$.

A result in stochastic analysis shows that a process with properties 1 & 2 must have the form [Kem17, Theorem 2.1]

$$W_t = \sqrt{\kappa} B_t + \alpha t,$$

where B is a one-dimensional Brownian motion, $\kappa \geq 0$, and $\alpha \in \mathbb{R}$. Property 3 then implies that $\alpha = 0$. Thus, it remains to determine the value of $\kappa \geq 0$. This comes from a computation in the proof of the scaling limit theorem, using properties of LERW. For other critical models we have analogous results with other values of κ (the parameter κ is, loosely speaking, determined by the universality class, or the conformal field theory associated to each critical model).

We only know from Loewner's theorem that taking $W_t = \sqrt{\kappa} B_t$ gives a family of growing sets $(K_t)_{t\geq 0}$ in the disc $\overline{\mathbb{D}}$. However, one can prove that these sets are generated by a curve in the following sense.

Definition 4.5. We say that growing sets $(K_t)_{t\geq 0}$ in the disc $\overline{\mathbb{D}}$ are *generated by a curve* if there exists a (continuous) curve $\gamma : [0,\infty) \to \overline{\mathbb{D}}$ such that for each time $t \geq 0$, the set $\mathbb{D} \setminus K_t$ is the connected component of $\mathbb{D} \setminus \gamma[0,t]$ containing the origin.

Theorem 4.6 (Brownian motion generates SLE curve)**.** *The growing sets $(K_t)_{t\geq 0}$ obtained from solving the Loewner equation* (LE) *with $W_t = \sqrt{\kappa} B_t$ are almost surely generated by a curve.*

Proof idea. For $\kappa \neq 8$, this was proven by Rohde & Schramm [RS05] by an elaborate argument relying on estimates for the derivative of the inverse conformal map g_t^{-1} near the driving function ζ_t. This estimate breaks down when $\kappa = 8$, but the result still holds (this is just a limitation of the proof). Lawler, Schramm & Werner proved that the case $\kappa = 8$ gives the scaling limit of the Peano curve for the uniform spanning tree, and the proof in particular implies that the limiting object is a curve. To date, there is no direct analytical proof[4] for the case of $\kappa = 8$. □

Definition 4.7. This random curve is called (radial) *Schramm-Loewner evolution*, SLE(κ). Its law is denoted $\mathbb{P}^{\mathbb{D};1,0}$. It is defined in $(\mathbb{D}, 1, 0)$ by the radial Loewner chain driven by $W = \sqrt{\kappa}B$, and in any other simply connected domain (D, y_0, x_0) with a boundary point $y_0 \in \partial D$ and an interior point $x_0 \in D$, the SLE(κ) law $\mathbb{P}^{D,y_0,x_0}$ is defined via conformal invariance as the pushforward $\varphi_* \mathbb{P}^{\mathbb{D};1,0}$ by the conformal map $\varphi : \mathbb{D} \to D$ sending 1 to $y_0 = \varphi(1)$ and 0 to $x_0 = \varphi(0)$.

Remark 4.8. Using properties of Bessel processes, one can show [RS05] that

- ▷ when $0 \leq \kappa \leq 4$, the SLE(κ) curve is simple;
- ▷ when $4 < \kappa < 8$, the SLE(κ) curve is not simple, nor space-filling;
- ▷ when $\kappa \geq 8$, the SLE(κ) curve is space-filling.

It is also important to note that (with the parameterization by $g_t'(0) = e^t$) the SLE(κ) curve does indeed reach it target point: $\lim_{t\to\infty} \gamma(t) =: \gamma(\infty) = 0$.

For convergence of curves, an appropriate topological space is the (Polish: complete separable metric) space $\mathcal{X}(D; x_0, y_0)$ of all unparameterized curves in $\overline{D}$ from x_0 to y_0 with metric

$$d_{\mathcal{X}}(\eta, \tilde{\eta}) := \inf_{\psi,\tilde{\psi}} \sup_{t\in[0,1]} |\eta(\psi(t)) - \tilde{\eta}(\tilde{\psi}(t))|,$$

where $\eta, \tilde{\eta} : [0, 1] \to \overline{D}$ are representatives of curves, and the infimum is taken over all reparameterizations, that is, increasing bijections $\psi, \tilde{\psi} : [0, 1] \to [0, 1]$.

Theorem 4.9 (Scaling limit of LERW is SLE(2))**.** *Let D be a simply connected domain. Consider LERW γ^δ in $D^\delta = D \cap \delta\mathbb{Z}^2$ started from x_0^δ and ending at $y_0^\delta \in \partial D^\delta$, where $x_0^\delta \to x_0 \in D$ and $y_0^\delta \to y_0 \in \partial D$ as $\delta \to 0$. Then, $\gamma^\delta \to \gamma$ in distribution (weakly as probability measures on the curve space $\mathcal{X}(D; x_0, y_0)$), where the law of γ is the radial SLE(2) in $\overline{D}$ from x_0 to y_0.*

We won't go into details of the convergence. Let's just make some comments:

- ▷ Radial SLE was introduced by Oded Schramm in 1999, and he also derived some properties that the limit of LERW, if exists, has to satisfy (e.g., the limit must be a simple curve). Theorem 4.9 was proven by Schramm with Lawler and Werner in 2001 [LSW04].
- ▷ The proof relies on properties of the underlying random walk and (discrete) harmonic functions. It has two main steps:
  1. Prove *precompactness*: there exists at least a subsequence that converges in $\mathcal{X}(D; x_0, y_0)$ (or in some other topological space). This step uses a priori estimates ruling out pathological behavior of the curves.[5]
  2. Show that all convergent subsequences must converge to the *same* limit. This step uses specific properties of the model (observable).

[4]There is a recent proof relying on the Gaussian free field [KMS21].

[5]In fact, the original proof goes in two steps: first establish precompactness for Loewner driving functions in $C([0,\infty), \mathbb{S}^1)$ with the uniform norm, and then strengthen the topology to the more geometric one in $\mathcal{X}(D; x_0, y_0)$ with additional (crossing) estimates.

5 Conformal restriction property for SLE(8/3)

For a given lattice model involving interfaces that should converge to conformally invariant curves satisfying the domain Markov property, guessing the value of κ may not be easy. Rigorously, the value comes from computations in the proofs, or sometimes there are special symmetries that can be used to identify κ. One such case is $\kappa = 8/3$, which conjecturally corresponds to scaling limits of self-avoiding walks (self-avoiding polymers). Let us consider self-avoiding nearest-neighbor walks on the discretization $\mathbb{D}^\delta := \mathbb{D} \times \delta\mathbb{Z}^2$ of the unit disc started from the origin and ending at the boundary point $1 \in \partial\mathbb{D}^\delta$. Since there are finitely many such walks, we can sample one at random with weight μ^{-N}, where N is the length of the walk and

$$\mu = \lim_{N\to\infty} \Big(\#\{\text{self-avoiding walks in } \mathbb{D}^\delta \text{ from } 0 \text{ to } 1 \text{ of length } N\} \Big)^{1/N}$$

is the so-called *connective constant* (on our case, associated to the square lattice). We will call this model the *self-avoiding walk (SAW)*. See the lecture notes [BDCGS12] for extensive discussion on self-avoiding walks (the lecture notes are ten years old, but there haven't been many advances in the understanding of the SAW model in the past decade).

Conjecture 5.1 (Scaling limit of SAW is SLE(8/3))**.** *The SAW random path γ^δ in $\mathbb{D}^\delta$ started from 0 and ending at 1 converges in distribution (weakly as probability measures on the curve space $\mathcal{X}(\mathbb{D};0,1)$) in the scaling limit $\delta \to 0$ to γ, the radial SLE(8/3) in $\overline{\mathbb{D}}$ from 0 to 1.*

To date, there is no proof even for that there would exist convergent subsequences for SAW in the curve space $\mathcal{X}(\mathbb{D};0,1)$, or in any other curve space with weaker topology — nor has it been shown that the driving functions of the curves γ^δ would converge even along a subsequence in $C([0,\infty),\mathbb{S}^1)$ to $\zeta = \exp(\mathrm{i}\sqrt{8/3}B)$. On random lattices though, a proof is known [GM21].

To support the conjecture, let us consider the "conformal restriction property", which is a natural property that the discrete SAW path satisfies, and which should survive in the scaling limit. It could be then used to perform the identification of the scaling limit, if one finds a convergent subsequence. As a reference, see the textbook [Law05, Sections 6.4–6.5], and for more discussion, e.g., Werner's lecture notes [Wer05] and references therein.

5.1 Obstacles and restriction property

Recall that one way to characterize the law of a random set K is by determining all probabilities that K doesn't intersect an obstacle A. More precisely, let $\mathcal{A}$ denote the collection of *obstacles*, that is, subsets $A \subset \overline{\mathbb{D}}$ such that A is compact, $A = \overline{A \cap \mathbb{D}}$ is compact, $\mathbb{D} \setminus A$ is simply connected, and $0, 1 \notin A$. For each obstacle A, let $\Phi_A : \mathbb{D} \setminus A \to \mathbb{D}$ be the unique conformal map such that $\Phi_A(0) = 0$ and $\Phi_A(1) = 1$.

Consider random sets $K \subset \overline{\mathbb{D}}$ such that $K \cap \partial\mathbb{D} = \{1\}$, $0 \in K$, K is connected, and $\mathbb{D} \setminus K$ is connected, and such that they satisfy the following restriction property:

Definition 5.2 (Conformal restriction property)**.** For any obstacle $A \in \mathcal{A}$, the law of $\Phi_A(K)$ conditioned on the event $\{K \cap A = \varnothing\}$ is the same as the law of K.

If one considers SAW on a discretization of the disc as above, this property is clearly satisfied. Furthermore, one can prove that SLE(8/3) is the *only* simple SLE(κ) curve that has the conformal restriction property. The chordal case was considered by Lawler, Schramm & Werner [LSW03], and the radial case is discussed, e.g., in Lawler's book [Law05]. Other simple SLE(κ) curves also behave relatively nicely under the maps Φ_A, but their laws become distorted by a factor that is related to the conformal anomaly in conformal field theory (see Section 5.3 below for a very brief discussion and Werner's lecture notes [Wer05, Chapter 5]).

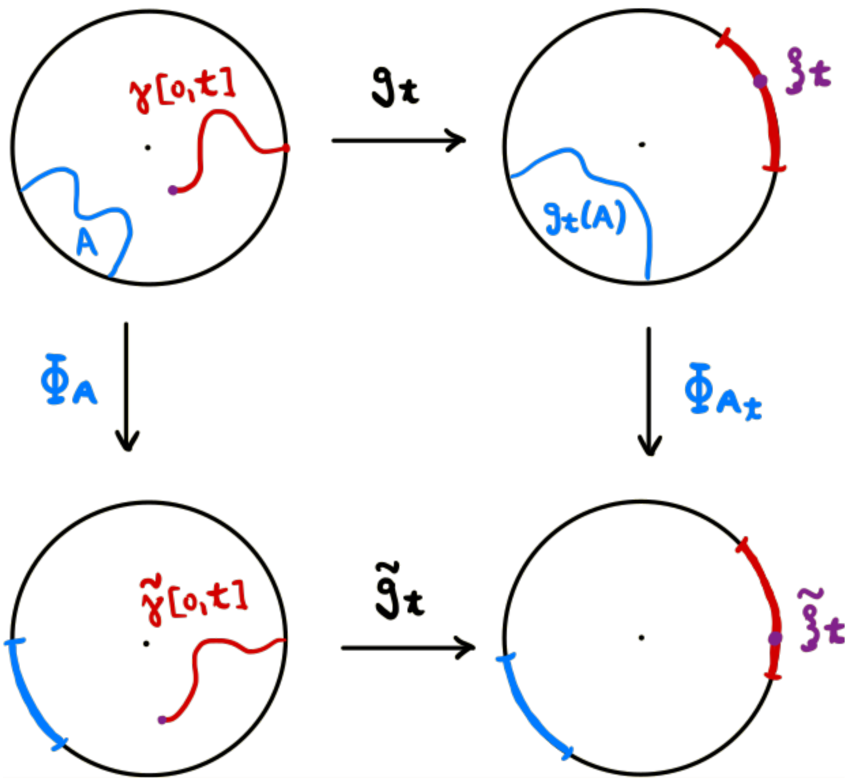


Figure 5.1: Illustration of the various maps: $\Phi_{A_t} \circ g_t = \tilde{g}_t \circ \Phi_A$. Here, $\zeta_t = e^{\mathrm{i}\sqrt{8/3}B_t}$.

5.2 Restriction property for SLE(8/3)

Fix an obstacle $A \in \mathcal{A}$. Let γ be an SLE(8/3) curve. We first want to study the probability of the event $\{\gamma \cap A = \varnothing\}$. To this end, consider the image $\tilde{\gamma} = \Phi_A(\gamma)$. Let

$$\tau_A := \inf\{t \geq 0 \mid \gamma(t) \in A\}$$

and write $\tilde{\gamma}[0,t] = \Phi_A(\gamma[0,t])$ when $t < \tau_A$. Consider the following maps (see Figure 5.1):

- ▷ Recall that the Loewner conformal map $g_t : \mathbb{D} \smallsetminus \gamma[0,t] \to \mathbb{D}$ is normalized as $g_t(0) = 0$ and $g_t'(0) = e^t > 0$.
- ▷ Let $\tilde{g}_t : \mathbb{D} \smallsetminus \tilde{\gamma}[0,t] \to \mathbb{D}$ be the conformal map such that $\tilde{g}_t(0) = 0$ and $\tilde{g}_t'(0) = e^t > 0$.
- ▷ Denote $A_t := g_t(A)$, and let $\Phi_{A_t} : \mathbb{D} \smallsetminus A_t \to \mathbb{D}$ be the conformal map such that $\Phi_{A_t} \circ g_t = \tilde{g}_t \circ \Phi_A$. See Figure 5.1.

We can investigate $\mathbb{P}[\gamma \cap A = \varnothing]$ by using a martingale. Recall that martingales are processes $(M_t)_{t\geq 0}$ such that, given the history up to time t, the conditional expectation of M observed at time $s \geq t$ equals the present value M_t. The conditional probability that γ avoids A if we explore γ up to time t is naturally a martingale. More precisely, the process

$$M_t := \mathbb{E}[\mathbb{1}\{\gamma \cap A = \varnothing\} \mid \gamma[0,t]]$$

is a bounded martingale by construction, its initial value is just the sought probability

$$M_0 = \mathbb{P}[\gamma \cap A = \varnothing],$$

and it converges almost surely as $t \to \infty$ to the random variable $\mathbb{1}\{\gamma \cap A = \varnothing\}$. Moreover, the conformal Markov property of γ shows that

$$M_t = \mathbb{P}^{\mathbb{D};1,0}[\gamma \cap A = \varnothing \mid \gamma[0,t]] = \mathbb{1}\{t < \tau_A\}\, \mathbb{P}^{\mathbb{D} \smallsetminus \gamma[0,t];\gamma(t),0}[\gamma \cap A = \varnothing]$$

$$= \mathbb{1}\{t < \tau_A\}\, \mathbb{P}^{\mathbb{D};1,0}[\gamma \cap (f_t(A)) = \varnothing].$$

(Here, $f_t : \mathbb{D} \smallsetminus \gamma[0,t] \to \mathbb{D}$ are the conformal maps normalized such that $f_t(0) = 0 = g_t(0)$, $f_t(\gamma(t)) = 1$, and $|f_t'(0)| = e^t = g_t'(0)$.) It turns out that the right-hand side can be expressed in terms of the derivative of the map Φ_{A_t}, and the formula is given in the next lemma.

Lemma 5.3. *For any $A \in \mathcal{A}$, the process*

$$M_t = |\Phi_{A_t}'(0)|^{5/48}\, |\Phi_{A_t}'(e^{\mathrm{i}\sqrt{8/3}B_t})|^{5/8}, \qquad t \in [0, \tau_A),$$

is a bounded martingale for SLE(8/3).

Proof idea. This is proven using Itô calculus. See [Law05, Section 6.5] for details. □

Proposition 5.4. *For any $A \in \mathcal{A}$, the radial SLE(8/3) curve γ satisfies*

$$\mathbb{P}[\gamma \cap A = \varnothing] = |\Phi_A'(0)|^{5/48}\, |\Phi_A'(1)|^{5/8}.$$

Proof idea. The left-hand side of the assertion equals

$$\mathbb{P}[\gamma \cap A = \varnothing] = \mathbb{P}[\tau_A = \infty],$$

while the right-hand side of the assertion equals

$$M_0 = |\Phi_A'(0)|^{5/48}\, |\Phi_A'(1)|^{5/8},$$

that is, the initial value of the bounded martingale M from Lemma 5.3. Moreover:

▷ If $\tau_A = \infty$, then using the transience of the SLE curve (i.e., that $\gamma(\infty) = 0$) one can prove that

$$\lim_{t\to\infty} \Phi_{A_t}'(e^{\mathrm{i}\sqrt{8/3}B_t}) = 1, \qquad \lim_{t\to\infty} \Phi_{A_t}'(0) = 1, \qquad \text{and thus} \qquad \lim_{t\to\infty} M_t = 1.$$

▷ If $\tau_A < \infty$, then one can prove that

$$\lim_{t\to\tau_A} \Phi_{A_t}'(e^{\mathrm{i}\sqrt{8/3}B_t}) = 0, \qquad \text{and thus} \qquad \lim_{t\to\tau_A} M_t = 0.$$

Thus, we obtain using the martingale convergence theorem that

$$\mathbb{P}[\gamma \cap A = \varnothing] = \mathbb{P}[\tau_A = \infty] = \mathbb{E}[M_{\tau_A}] = \mathbb{E}[M_0] = M_0,$$

which is what we sought to prove. □

From the explicit formula for $\mathbb{P}[\gamma \cap A = \varnothing]$ from Proposition 5.4, we can verify the conformal restriction property for SLE(8/3).

Theorem 5.5 (Conformal restriction property of SLE(8/3))**.** *The radial SLE(8/3) curve γ satisfies the property that, for any obstacle $A \in \mathcal{A}$, the law of $\Phi_A(\gamma)$ conditioned on the event $\{\gamma \cap A = \varnothing\}$ is the same as the law of γ.*

Proof idea. Fix $A \in \mathcal{A}$. Conditioned on the event $\{\gamma \cap A = \varnothing\}$, consider another obstacle $C \in \mathcal{A}$ for the curve $\tilde{\gamma} = \Phi_A(\gamma)$. Using conformal invariance of SLE(8/3) and the formula from the previous theorem, we have

$$\begin{aligned}
\mathbb{P}[\tilde{\gamma} \cap C = \varnothing \mid \gamma \cap A = \varnothing] &= \frac{\mathbb{P}[\gamma \cap \Phi_A^{-1}(C) = \varnothing]}{\mathbb{P}[\gamma \cap A = \varnothing]} \\
&= \frac{\mathbb{P}[\gamma \cap (\mathbb{D} \smallsetminus (\Phi_A^{-1} \circ \Phi_C^{-1})(\mathbb{D})) = \varnothing]}{\mathbb{P}[\gamma \cap A = \varnothing]} \\
&= \frac{|\Phi_A'(0)|^{5/48}\, |\Phi_C'(0)|^{5/48}\, \Phi_A'(1)^{5/8}\, \Phi_C'(1)^{5/8}}{|\Phi_A'(0)|^{5/48}\, \Phi_A'(1)^{5/8}} \\
&= |\Phi_C'(0)|^{5/48}\, \Phi_C'(1)^{5/8} \\
&= \mathbb{P}[\gamma \cap C = \varnothing].
\end{aligned}$$

This proves the conformal restriction property. □

5.3 Restriction property fails for other simple SLE(κ) curves

Let now γ be an SLE(κ) curve for some $\kappa \in (0,4]$. Let's consider the event $\{\gamma \cap A = \varnothing\}$ in this more general case, using the same maps as before (Figure 5.1).

Remark 5.6. For $\kappa \neq 8/3$, the martingale M of Lemma 5.3 includes a non-trivial correction term that doesn't behave well in the calculation used in the proof of Theorem 5.5. Namely, the process

$$M_t = |\Phi'_{A_t}(0)|^{i(\kappa)} \, |\Phi'_{A_t}(e^{\mathrm{i}\sqrt{\kappa}B_t})|^{h(\kappa)} \exp\Big(- \frac{c(\kappa)}{6} \int_0^t S\Phi_{A_s}(e^{\mathrm{i}\sqrt{\kappa}B_s}) \, \mathrm{d}s\Big), \qquad t \in [0, \tau_A),$$

where[6]

$$i(\kappa) = \frac{(6-\kappa)(\kappa-2)}{8\kappa}, \qquad h(\kappa) = \frac{6-\kappa}{2\kappa}, \qquad c(\kappa) = \frac{(3\kappa-8)(6-\kappa)}{2\kappa},$$

and where $S\Phi_{A_s}$ is the *Schwarzian derivative* defined as

$$Sf(z) = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\Big(\frac{f''(z)}{f'(z)}\Big)^2,$$

is a martingale (bounded when $\kappa \leq 8/3$, that is, $c(\kappa) \leq 0$). One can also show [Law05, Proposition 5.22] that the term involving the Schwarzian derivative can alternatively be written in terms of the *Brownian loop measure* $\mu_{\mathbb{D}}^{\mathrm{loop}}$ introduced by Lawler, Schramm & Werner [LSW03, LW04][7]:

$$-\frac{1}{3} \int_0^t S\Phi_{A_s}(e^{\mathrm{i}\sqrt{\kappa}B_s}) \, \mathrm{d}s = \mu_{\mathbb{D}}^{\mathrm{loop}}(\gamma[0,t], A)$$

is the measure of those Brownian loops in $\mathbb{D}$ that intersect both sets $\gamma[0,t]$ and A. This term is responsible of the failure of the conformal restriction property for simple SLE(κ) curves with $\kappa \neq 8/3$. See Werner's lecture notes [Wer05] for more details.

Proposition 5.7 (Boundary perturbation for SLE)**.** *Let $\kappa \in (0,4]$. Fix an obstacle $A \in \mathcal{A}$. Then, radial SLE(κ) in $\mathbb{D} \smallsetminus A$ from 1 to 0 is absolutely continuous with respect to radial SLE(κ) in $\mathbb{D}$ from 1 to 0, with Radon-Nikodym derivative given by*

$$\begin{aligned} \frac{\mathrm{d}\mathbb{P}^{\mathbb{D}\smallsetminus A;1,0}}{\mathrm{d}\mathbb{P}^{\mathbb{D};1,0}}(\gamma) &= \mathbb{1}\{\gamma \cap A = \varnothing\} \, \frac{\exp\big(- \frac{c(\kappa)}{6} \int_0^\infty S\Phi_{A_s}(e^{\mathrm{i}\sqrt{\kappa}B_s}) \, \mathrm{d}s\big)}{|\Phi'_A(0)|^{i(\kappa)} \, |\Phi'_A(1)|^{h(\kappa)}} \\ &= \mathbb{1}\{\gamma \cap A = \varnothing\} \, \frac{\exp\big(\frac{c(\kappa)}{2} \, \mu_{\mathbb{D}}^{\mathrm{loop}}(\gamma[0,\infty), A)\big)}{|\Phi'_A(0)|^{i(\kappa)} \, |\Phi'_A(1)|^{h(\kappa)}}. \end{aligned}$$

Proof idea. This is a well-known property in the chordal case, see for example [KL07, Proposition 3.1]. The same arguments can be used in the radial case. □

6 Multiple SLE and applications to critical Ising model

Using the generalization of conformal restriction from Proposition 5.7, one gets an idea for constructing "multiple SLEs", which are candidates for scaling limits of multiple interfaces in critical models — e.g., those appearing in Figure 1.2(right).

[6] The number $h(\kappa)$ is the "boundary exponent" and $i(\kappa)$ is the "interior exponent" [Law09]. The number $c(\kappa)$ is the central charge of the corresponding conformal field theory.

[7] The difference of 1/2 compared to [BPW21] is due to normalization conventions for Brownian loop measure.

6.1 Multichordal SLE

Let $D \subsetneq \mathbb{C}$ be a simply connected Jordan domain with $2N$ distinct points $x_1, x_2, \ldots, x_{2N} \in \partial D$ appearing in counterclockwise order along the boundary (called *topological polygon*). We consider curves $\overline{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_N)$ in D each of which connects two points among $\{x_1, x_2, \ldots, x_{2N}\}$. These curves can have various planar (non-crossing) connectivities, described in terms of planar pair partitions (planar *link patterns*), that we write in the form

$$\alpha = \{\{a_1, b_1\}, \ldots, \{a_N, b_N\}\} \in \mathrm{LP}_N,$$

where $\{a_1, b_1, \ldots, a_N, b_N\} = \{1, 2, \ldots, 2N\}$, and where LP_N denotes the set of all such planar link patterns. Note that for each fixed $N \in \mathbb{N}$, the total number of planar link patterns is the *Catalan number* $C_N = \frac{1}{N+1}\binom{2N}{N} = \#\mathrm{LP}_N$. An example of (discrete) curves $\overline{\gamma}$ is obtained from interfaces in the critical Ising model with alternating boundary conditions at the points $x_1, x_2, \ldots, x_{2N}$, see Figure 1.2. We wish to describe the scaling limits of these interfaces using SLE type curves.

Since the interfaces connect boundary-to-boundary (i.e., they are chords in the domain D), we consider the *chordal* SLE(κ) obtained from the *chordal Loewner chain*, that is, the solution $(g_t)_{t \geq 0}$ to the chordal Loewner equation given for each $z \in \overline{\mathbb{H}} = \{z \in \mathbb{C} \mid \mathfrak{Im}(z) \geq 0\}$ as

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}, \qquad g_0(z) = z,$$

where $W_t = \sqrt{\kappa} B_t$, with B being the one-dimensional Brownian motion started at $B_0 = 0$. Similarly as for Theorem 4.3, $(g_t(z))_{t \in [0, T_z)}$ is defined up to the swallowing time of the point z,

$$T_z := \sup\{t \geq 0 \mid \liminf_{s \nearrow t} |g_s(z) - W_s| > 0\} \in [0, +\infty]$$

and $g_t : \mathbb{H} \smallsetminus K_t \to \mathbb{H}$ are conformal maps defined off $K_t = \{z \in \overline{\mathbb{H}} \mid T_z \leq t\}$. They satisfy the normalization $\lim_{z \to \infty} |g_t(z) - z| = 0$, which determines them uniquely.

Analogously to Theorem 4.6, one can show [RS05] that for the driving function $W_t = \sqrt{\kappa} B_t$, the growing sets $(K_t)_{t \geq 0}$ are generated by a curve $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ such that for each time $t \geq 0$, the set $\mathbb{H} \smallsetminus K_t$ is the unbounded connected component of $\mathbb{H} \smallsetminus \gamma[0, t]$. One can also prove that the curve is transient, that is, $\lim_{t \to \infty} \gamma(t) =: \gamma(\infty) = \infty$, which is a boundary point of $\mathbb{H}$.

Definition 6.1. This random curve is called (chordal) *Schramm-Loewner evolution*, SLE(κ). Its law is denoted $\mathbb{P}^{\mathbb{H}; 0, \infty}$. It is defined in $(\mathbb{H}; 0, \infty)$ by the chordal Loewner chain driven by $W = \sqrt{\kappa} B$, and in any other simply connected domain (D, x, y) with two distinct boundary points $x, y \in \partial D$, the SLE(κ) law $\mathbb{P}^{D, x, y}$ is defined via conformal invariance as the pushforward $\varphi_* \mathbb{P}^{\mathbb{H}; 0, \infty}$ by any conformal map $\varphi : \mathbb{H} \to D$ sending 0 to $x = \varphi(0)$ and ∞ to $y = \varphi(\infty)$. (One can check that the law $\mathbb{P}^{\mathbb{H}; 0, \infty}$ is scale-invariant, which implies that the choice of φ doesn't matter.)

- ▷ Let $\mathcal{X}^0(D; x, y)$ denote the set of continuous simple unparameterized curves in $\overline{D}$ connecting $x \in \partial D$ and $y \in \partial D$ such that they only touch the boundary in $\{x, y\}$. When $\kappa \leq 4$, the chordal SLE(κ) curve belongs to this space almost surely [RS05, Zha08][8].
- ▷ For each link pattern $\alpha \in \mathrm{LP}_N$, let $\mathcal{X}_\alpha(D; x_1, \ldots, x_{2N})$ denote the set of families $\overline{\gamma} = (\gamma_1, \ldots, \gamma_N)$ of pairwise disjoint curves such that $\gamma_j \in \mathcal{X}^0(D; x_{a_j}, x_{b_j})$ for all $j \in \{1, 2, \ldots, N\}$.

[8]Here we also need the *reversibility property* of chordal SLE: the SLE(κ) curve from x to y has the same law as the SLE(κ) curve from y to x. This property is very non-trivial to prove. A proof for the case $\kappa \leq 4$ was first obtained by Zhan [Zha08].

Definition 6.2 (Multichordal SLE, i.e., N-SLE)**.** Let $\kappa \in (0,4]$. For $N \geq 2$ and for any link pattern $\alpha \in \mathrm{LP}_N$, we call a probability measure on the families $(\gamma_1, \ldots, \gamma_N) \in \mathcal{X}_\alpha(D; x_1, \ldots, x_{2N})$ an N-SLE(κ) *associated to* α if for each $j \in \{1, 2, \ldots, N\}$, the conditional law of the curve γ_j given $\{\gamma_1, \gamma_2, \ldots, \gamma_N\} \smallsetminus \{\gamma_j\}$ is the chordal SLE(κ) connecting x_{a_j} and x_{b_j} in the connected component of the domain $D \smallsetminus \bigcup_{i \neq j} \gamma_i$ containing the endpoints x_{a_j} and x_{b_j} of γ_j on its boundary.

Theorem 6.3. *Let $\kappa \in (0,4]$. For any polygon $(D; x_1, \ldots, x_{2N})$ and link pattern $\alpha \in \mathrm{LP}_N$, there exists a unique N-SLE(κ) associated to α.*

Proof idea. This is proven (in full generality) in [BPW21]. The idea is a Markov chain coupling argument, sampling at each step one curve from its conditional law given the other curves. One can prove that such a Markov chain on the curve space $\mathcal{X}_\alpha(D; x_1, \ldots, x_{2N})$ has a unique stationary measure, that is the N-SLE(κ) associated to α. □

Multichordal SLE(κ) is a family of SLE(κ) curves with interaction. It can be constructed as follows [Law09]. Let $\mathbb{Q}_\alpha^\kappa$ denote the product measure of N independent chordal SLE(κ) curves associated to the link pattern α. Denote by $\mathbb{E}\mathbb{Q}_\alpha^\kappa$ the expectation with respect to $\mathbb{Q}_\alpha^\kappa$. Then, the multichordal N-SLE(κ) probability measure $\mathbb{P}_\alpha^\kappa$ on $\mathcal{X}_\alpha(D; x_1, \ldots, x_{2N})$ can be obtained by weighting $\mathbb{Q}_\alpha^\kappa$ with the Radon-Nikodym derivative[9]

$$R_\alpha^\kappa(\overline{\gamma}) = \frac{\mathrm{d}\mathbb{P}_\alpha^\kappa}{\mathrm{d}\mathbb{Q}_\alpha^\kappa}(\overline{\gamma}) := \frac{\mathbb{1}\{\gamma_i \cap \gamma_j = \varnothing \ \forall i \neq j\} \ \exp\Big(\frac{c(\kappa)}{2}\, m_D(\overline{\gamma})\Big)}{\mathbb{E}\mathbb{Q}_\alpha^\kappa\Big[\mathbb{1}\{\gamma_i \cap \gamma_j = \varnothing \ \forall i \neq j\} \ \exp\Big(\frac{c(\kappa)}{2}\, m_D(\overline{\gamma})\Big)\Big]}, \tag{6.1}$$

where $c(\kappa) = \frac{(3\kappa-8)(6-\kappa)}{2\kappa}$ and $m_D(\overline{\gamma})$ is expressed in terms of the Brownian loop measure μ_D^{loop}: $m_D(\gamma^\kappa) := 0$ if $N = 1$, and

$$m_D(\overline{\gamma}) := \sum_{p=2}^{N} \mu_D^{\mathrm{loop}}\Big(\big\{\ell \,\big|\, \ell \cap \gamma_i \neq \varnothing \text{ for at least } p \text{ of the } i \in \{1, \ldots, N\}\big\}\Big) \qquad \text{if } N \geq 2.$$

One can prove that this is a conformally invariant quantity. See [Law09] for more details.

Definition 6.4. The *pure partition functions* of multiple SLE(κ) are defined in terms of the total mass of the N-SLE(κ) measure:

$$\mathcal{Z}_\alpha(D; x_1, \ldots, x_{2N}) := \Big(\prod_{j=1}^{N} P_D(x_{a_j}, x_{b_j})\Big)^{h(\kappa)} \times \mathbb{E}\mathbb{Q}_\alpha^\kappa\Big[\mathbb{1}\{\gamma_i \cap \gamma_j = \varnothing \ \forall i \neq j\} \ \exp\Big(\frac{c(\kappa)}{2}\, m_D(\overline{\gamma})\Big)\Big],$$

where $h(\kappa) = \frac{6-\kappa}{2\kappa}$ is the boundary exponent, $P_D(x,y)$ is the boundary Poisson kernel, that is the unique conformally covariant function defined as

$$P_D(x,y) := |\varphi'(x)|\,|\varphi'(y)|\, P_{\mathbb{H}}(\varphi(x), \varphi(y)), \qquad \text{where} \qquad P_{\mathbb{H}}(z,w) := |w-z|^{-2},$$

and where $\varphi \colon D \to \mathbb{H}$ is any conformal map.

The motivation for this definition comes, roughly, from considering chordal SLE(κ) as a unnormalized (non-probability) measure on $\mathcal{X}^0(D; x, y)$, which satisfies the following analogue of the boundary perturbation property [KL07, Proposition 3.1] (cf. Proposition 5.7):

Proposition 6.5 (Boundary perturbation for chordal SLE)**.** *Let $\kappa \in (0,4]$. Fix a compact set $A \subset \overline{D}$ such that $A = \overline{A \cap D}$, the domain $D \smallsetminus A$ is simply connected, and $x, y \notin A$. Then, chordal*

[9] The difference of 1/2 compared to [BPW21] is due to normalization conventions for Brownian loop measure.

SLE(κ) in $D \smallsetminus A$ from x to y is absolutely continuous with respect to chordal SLE(κ) in D from x to y, with Radon-Nikodym derivative given by

$$\frac{\mathrm{d}\mathbb{P}^{D\smallsetminus A;x,y}}{\mathrm{d}\mathbb{P}^{D;x,y}}(\gamma) = \mathbb{1}\{\gamma \cap A = \varnothing\} \, \frac{\exp\big(\frac{c(\kappa)}{2}\,\mu_D^{\mathrm{loop}}(\gamma[0,\infty),A)\big)}{|\Phi_A'(x)|^{h(\kappa)}|\Phi_A'(y)|^{h(\kappa)}}, \tag{6.2}$$

where $\Phi_A : D \smallsetminus A \to D$ is any conformal map fixing x and y.

Indeed, using the definition of the Poisson kernel in (6.2), we have

$$\begin{aligned}\frac{\mathrm{d}\mathbb{P}^{D\smallsetminus A;x,y}}{\mathrm{d}\mathbb{P}^{D;x,y}}(\gamma) &= \mathbb{1}\{\gamma \cap A = \varnothing\} \, \frac{\exp\big(\frac{c(\kappa)}{2}\,\mu_D^{\mathrm{loop}}(\gamma[0,\infty),A)\big)}{|\Phi_A'(x)|^{h(\kappa)}|\Phi_A'(y)|^{h(\kappa)}} \\ &= \mathbb{1}\{\gamma \cap A = \varnothing\} \left(\frac{P_D(x,y)}{P_{D\smallsetminus A}(x,y)}\right)^{h(\kappa)} \exp\Big(\frac{c(\kappa)}{2}\,\mu_D^{\mathrm{loop}}(\gamma[0,\infty),A)\Big)\end{aligned}$$

and rearranging this, we obtain

$$\big(P_{D\smallsetminus A}(x,y)\big)^{h(\kappa)}\,\mathrm{d}\mathbb{P}^{D\smallsetminus A;x,y}(\gamma) = \big(P_D(x,y)\big)^{h(\kappa)}\,\exp\Big(\frac{c(\kappa)}{2}\,\mu_D^{\mathrm{loop}}(\gamma[0,\infty),A)\Big)\,\mathrm{d}\mathbb{P}^{D;x,y}(\gamma).$$

The total mass of the right-hand side appears as an ingredient in the construction of $\mathcal{Z}_\alpha$. An inclusion-exclusion argument gives the loop measure term $m_D(\overline{\gamma})$, see [Law09, PW19, BPW21].

Remark 6.6. Note that when $\kappa < 8/3$, we have $c(\kappa) < 0$, so the normalization factor in (6.1) is clearly finite. Write

$$\Phi_\kappa(\overline{\gamma}) := \frac{\kappa\, c(\kappa)}{2}\, m_D(\overline{\gamma}).$$

Then, for all $\overline{\gamma} \in \mathcal{X}_\alpha(D; x_1, \ldots, x_{2N})$, as $\kappa \searrow 0$, we have

$$\Phi_\kappa(\overline{\gamma}) = -\frac{c(\kappa)\kappa}{24}\Phi_0(\overline{\gamma}) \quad \searrow \quad \Phi_0(\overline{\gamma}) = -12 m_D(\overline{\gamma}) < 0,$$

since $-c(\kappa)\kappa \nearrow 24$. This observation is useful when studying large deviations as $\kappa \searrow 0$ later on.

Using the function Φ_κ, we can write the Radon-Nikodym derivative (6.1) in the form

$$R_\alpha^\kappa(\overline{\gamma}) = \frac{\exp\big(\frac{1}{\kappa}\,\Phi_\kappa(\overline{\gamma})\big)}{\mathbb{E}\mathbb{Q}_\alpha^\kappa\big[\exp\big(\frac{1}{\kappa}\,\Phi_\kappa(\overline{\gamma})\big)\big]},$$

which looks sort of like a Bolzmann measure on curves.

6.2 Crossing probabilities in critical Ising model

Just like for a single interface (converging to SLE), one can also prove, or conjecture, that multiple interfaces in critical lattice models converge in the scaling limit to multiple SLE. As an example, let us look at the critical Ising model with alternating boundary conditions, having several "+" and "−" segments on the boundary (see Figure 1.2). Let $x_1^\delta, \ldots, x_{2N}^\delta$ be the boundary points where the boundary conditions change between "+" and "−". There are several macroscopic boundary-to-boundary interfaces $(\gamma_1^\delta, \ldots, \gamma_N^\delta)$. They may connect the boundary points in various ways: for each $\alpha \in \mathrm{LP}_N$, there is a positive chance that $(\gamma_1^\delta, \ldots, \gamma_N^\delta)$ have the connectivity α. We are interested in the scaling limits of the interfaces and their connection probabilities.

To fix the setup, write $D^\delta := D \cap \delta\mathbb{Z}^2$ and suppose that $x_j^\delta \to x_j$ for each j. For the topology on the curve space $\mathcal{X}_\alpha(\mathbb{D}; x_1, \ldots, x_{2N})$, we can just take

$$d_{\mathcal{X}}\big((\eta_1, \ldots, \eta_N), (\tilde{\eta}_1, \ldots, \tilde{\eta}_N)\big) := \max_{1\le j\le N} d_{\mathcal{X}}(\eta_j, \tilde{\eta}_j).$$

Theorem 6.7. *The following hold for the critical Ising model on $(D^\delta; x_1^\delta, \ldots, x_{2N}^\delta)$ with alternating boundary conditions:*

1. [PW23, Theorem 1.1] *For each link pattern $\alpha \in \mathrm{LP}_N$ the following convergence holds:*

$$\lim_{\delta \to 0} \mathbb{P}^\delta[\text{ interfaces } (\gamma_1^\delta, \ldots, \gamma_N^\delta) \text{ form connectivity } \alpha\,] = \frac{\mathcal{Z}_\alpha(D; x_1, \ldots, x_{2N})}{\sum_{\beta \in \mathrm{LP}_N} \mathcal{Z}_\beta(D; x_1, \ldots, x_{2N})}, \tag{6.3}$$

where $\{\mathcal{Z}_\alpha \mid \alpha \in \mathrm{LP}_N\}$ are the pure partition functions of multiple SLE(κ) with $\kappa = 3$.

2. [BPW21, Proposition 1.3]

   ▷ *Let $\alpha \in \mathrm{LP}_N$. Then, as $\delta \to 0$, conditionally on the event that they form the connectivity α, the law of the collection of critical Ising interfaces converges weakly to the N-SLE(3) associated to α.*

   ▷ *In particular, as $\delta \to 0$, the law of a single curve in this collection connecting two points x_j and $x_{\alpha(j)}$, where $\{j, \alpha(j)\} \in \alpha$, converges weakly to a conformal image of the Loewner chain whose driving function W in the upper half-plane $\mathbb{H}$ is given by the SDEs*

$$\begin{cases} \mathrm{d}W_t = \sqrt{3}\,\mathrm{d}B_t + 3\partial_j \log \mathcal{Z}_\alpha\big(\mathbb{H}; g_t(x_1), \ldots, g_t(x_{j-1}), W_t, g_t(x_{j+1}), \ldots, g_t(x_{2N})\big)\,\mathrm{d}t, \\ \mathrm{d}g_t(x_i) = \frac{2\,\mathrm{d}t}{g_t(x_i) - W_t}, \qquad \text{for } i \neq j, \end{cases}$$

   with initial values $W_0 = x_j$, and $g_0(x_i) = x_i$ for $i \neq j$.

3. [Izy15, Theorem 3.1], [Izy17, Theorem 1.1], and [PW23, Theorem 4.1 & Proposition 5.1] *As $\delta \to 0$, the law of a single curve in the collection of critical Ising interfaces starting from x_j converges weakly to a conformal image of the Loewner chain whose driving function W in the upper half-plane $\mathbb{H}$ is given by the SDEs*

$$\begin{cases} \mathrm{d}W_t = \sqrt{3}\,\mathrm{d}B_t + 3\partial_j \log\Big(\sum_{\beta \in \mathrm{LP}_N} \mathcal{Z}_\beta\big(\mathbb{H}; g_t(x_1), \ldots, g_t(x_{j-1}), W_t, g_t(x_{j+1}), \ldots, g_t(x_{2N})\big)\Big)\,\mathrm{d}t, \\ \mathrm{d}g_t(x_i) = \frac{2\,\mathrm{d}t}{g_t(x_i) - W_t}, \qquad \text{for } i \neq j, \end{cases} \tag{6.4}$$

with initial values $W_0 = x_j$, and $g_0(x_i) = x_i$ for $i \neq j$. This curve terminates almost surely at one of the marked points x_ℓ, where ℓ has different parity than j.

Remark 6.8. In the first nontrivial case of $N = 2$, the crossing formula (6.3) was predicted by Arguin & Saint-Aubin [ASA02] (this is an analogue of Cardy's formula for critical Bernoulli percolation): the pure partition functions in this case are given by

$$\mathcal{Z}_{\{\{1,4\},\{2,3\}\}}(\mathbb{H}; x_1, x_2, x_3, x_4) = \frac{2\Gamma(4/3)}{\Gamma(8/3)\Gamma(5/3)} \left(\frac{(x_2 - x_1)(x_4 - x_3)}{(x_4 - x_2)(x_3 - x_1)}\right)^{2/3} \frac{{}_2F_1\left(\frac{4}{3}, -\frac{1}{3}, \frac{8}{3}; \frac{(x_2 - x_1)(x_4 - x_3)}{(x_4 - x_2)(x_3 - x_1)}\right)}{(x_4 - x_1)(x_3 - x_2)}$$

$$\mathcal{Z}_{\{\{1,2\},\{3,4\}\}}(\mathbb{H}; x_1, x_2, x_3, x_4) = \frac{2\Gamma(4/3)}{\Gamma(8/3)\Gamma(5/3)} \left(\frac{(x_4 - x_1)(x_3 - x_2)}{(x_4 - x_2)(x_3 - x_1)}\right)^{2/3} \frac{{}_2F_1\left(\frac{4}{3}, -\frac{1}{3}, \frac{8}{3}; \frac{(x_4 - x_1)(x_3 - x_2)}{(x_4 - x_2)(x_3 - x_1)}\right)}{(x_2 - x_1)(x_4 - x_3)},$$

for $x_1 < x_2 < x_3 < x_4$, where $\frown\frown = \{\{1,2\},\{3,4\}\}$ and (nested arcs) $= \{\{1,4\},\{2,3\}\}$ are the two possible link patterns. These formulas and certain other special cases appear in [BBK05, Izy15]. In general, explicit formulas for the probability amplitudes $\mathcal{Z}_\alpha$ are not known when $\kappa = 3$.

Proof idea for items 2 & 3. The convergence of one critical Ising interface ($N = 1$) was summarized in [CDCH$^+$14], based on Smirnov's ideas. This is established in two steps.

▷ First, one proves that the sequence $(\gamma^\delta)_{\delta>0}$ of lattice interfaces on D^δ is relatively compact in the space $\mathcal{X}$ of curves. Thus, one deduces that there exist convergent subsequences as $\delta \to 0$. For the Ising model, the relative compactness is established using topological crossing estimates, see in particular [KS17].

▷ Second, one has to prove that all of the subsequences in fact converge to a unique limit, identified as the chordal SLE(κ) with $\kappa = 3$. For the identification of the limit, Smirnov used a discrete holomorphic martingale observable [Smi06, Smi10], that is, a solution to a discrete boundary value problem on D^δ, converging as $\delta \to 0$ to the solution of the corresponding boundary value problem on D. Using the martingale observable, he identified the Loewner driving function of the scaling limit curve as $\sqrt{3}B_t$.

For multiple curves, the relative compactness follows from the one-curve case [Kar19, Wu20]. For the identification, one can use either a multipoint discrete holomorphic observable, as was done for item 3 in [Izy15, Izy17], or the classification of multiple SLE probability measures by the Markov chain argument done for item 2 in [BPW21]. □

Proof idea for item 1. Denote the random connectivity of the interfaces $(\gamma_1^\delta, \ldots, \gamma_N^\delta)$ by ϑ^δ. We prove the claim by induction on $N \geq 1$ and using a standard martingale argument in SLE theory. The claim is trivial for $N = 1$ because both sides of (6.3) equal one. Thus, we assume that the claim holds for $N-1$, fix $\alpha \in \mathrm{LP}_N$ and aim to prove the claim for $\mathbb{P}^\delta[\vartheta^\delta = \alpha]$. The probabilities $(\mathbb{P}^\delta[\vartheta^\delta = \alpha])_{\delta>0}$ form a sequence of numbers in $[0,1]$, so there is always a subsequential limit. To show (6.3), it hence suffices to prove that

$$\lim_{n\to\infty} \mathbb{P}^{\delta_n}[\vartheta^{\delta_n} = \alpha] = \frac{\mathcal{Z}_\alpha(D; x_1, \ldots, x_{2N})}{\sum_{\beta\in\mathrm{LP}_N} \mathcal{Z}_\beta(D; x_1, \ldots, x_{2N})}$$

for any convergent subsequence.

Note that the link pattern α contains at least one link of type $\{j, j+1\}$. For definiteness, we assume that $j = 1$, so $\{1,2\} \in \alpha$. Let us also assume that $D = \mathbb{H}$. Item 3 says that the curve $\gamma_1^{\delta_n}$ converges (weakly) to γ_1, given by the Loewner chain whose driving function satisfies the SDEs (6.4). We may couple the curves (by the Skorohod representation theorem) into the same probability space so that they converge almost surely.

First, let us analyze the limit curve γ_1. One can prove that the ratio

$$M_t := \frac{\mathcal{Z}_\alpha(W_t, g_t(x_2), \ldots, g_t(x_{2N}))}{\sum_{\beta\in\mathrm{LP}_N} \mathcal{Z}_\beta(W_t, g_t(x_2), \ldots, g_t(x_{2N}))}$$

is a bounded martingale. Let us consider the limit of M_t as

$$t \to T := \min_{i\neq 1} T_{x_i} \;\in\; (0, +\infty],$$

upon the first time when the curve γ_1 swallows one of the other marked points (that is, the blow-up time of the Loewner equation). Denote by H_{γ_1} the unbounded connected component of $\mathbb{H} \smallsetminus \gamma_1[0,T]$, and by $\hat{\alpha} = \alpha \smallsetminus \{1,2\} \in \mathrm{LP}_{N-1}$ the sought connectivity of the remaining curves $(\gamma_2, \ldots, \gamma_N)$.

▷ On the (good) event $\{\gamma_1(T) = x_2\}$, one can prove that almost surely,

$$\begin{aligned} M_t &= \left(\frac{\mathcal{Z}_\alpha(W_t, g_t(x_2), \ldots, g_t(x_{2N}))}{\mathcal{Z}_{\frown}(W_t, g_t(x_2))} \right) \left(\frac{\mathcal{Z}_{\frown}(W_t, g_t(x_2))}{\sum_{\beta\in\mathrm{LP}_N} \mathcal{Z}_\beta(W_t, g_t(x_2), \ldots, g_t(x_{2N}))} \right) \\ &\xrightarrow{t\to T} \frac{\mathcal{Z}_{\hat{\alpha}}(g_T(x_3), \ldots, g_T(x_{2N}))}{\sum_{\beta\in\mathrm{LP}_{N-1}} \mathcal{Z}_\beta(g_T(x_3), \ldots, g_T(x_{2N}))} = \frac{\mathcal{Z}_{\hat{\alpha}}(H_{\gamma_1}; x_3, \ldots, x_{2N})}{\sum_{\beta\in\mathrm{LP}_{N-1}} \mathcal{Z}_\beta(H_{\gamma_1}; x_3, \ldots, x_{2N})}. \end{aligned}$$

▷ On the other hand, for each $\ell \in \{2,3,\ldots,N\}$, on the (bad) event $\{\gamma_1(T) = x_{2\ell}\}$, one can show that almost surely

$$M_t = \frac{\mathcal{Z}_\alpha(W_t, g_t(x_2), \ldots, g_t(x_{2N}))}{\sum_{\beta \in \mathrm{LP}_N} \mathcal{Z}_\beta(W_t, g_t(x_2), \ldots, g_t(x_{2N}))} \quad \overset{t \to T}{\longrightarrow} \quad 0.$$

Applying the optional stopping theorem to the bounded martingale M gives the identity

$$\begin{aligned} M_0 &= \frac{\mathcal{Z}_\alpha(x_1, \ldots, x_{2N})}{\sum_{\beta \in \mathrm{LP}_N} \mathcal{Z}_\beta(x_1, \ldots, x_{2N})} \\ &= \mathbb{E}\bigg[\mathbb{1}\{\gamma_1(T) = x_2\} \frac{\mathcal{Z}_{\hat{\alpha}}(H_{\gamma_1}; x_3, \ldots, x_{2N})}{\sum_{\beta \in \mathrm{LP}_{N-1}} \mathcal{Z}_\beta(H_{\gamma_1}; x_3, \ldots, x_{2N})} \bigg] = \mathbb{E}[M_T]. \end{aligned} \tag{6.5}$$

Next, let us consider the discrete interface $\gamma_1^{\delta_n}$. On the event $\{\gamma_1^{\delta_n}(T^{\delta_n}) = x_2^{\delta_n}\}$, we denote by H^{δ_n} the connected component of of the complement of $\gamma_1^{\delta_n}$ with $x_3^{\delta_n}, \ldots, x_{2N}^{\delta_n}$ on its boundary. Using the domain Markov property of the Ising model, the induction hypothesis, and the conformal invariance of the right-hand side of (6.3) and the conformal invariance of the SLE(3) type curve γ, we find that

$$\begin{aligned} \mathbb{E}^{\delta_n}[\mathbb{1}\{\vartheta^{\delta_n} = \alpha\} \,|\, \gamma_1^{\delta_n}] &= \mathbb{1}\{\gamma_1^{\delta_n}(T^{\delta_n}) = x_2^{\delta_n}\}\, \mathbb{E}^{\delta_n}[\mathbb{1}\{\widehat{\vartheta}^{\delta_n} = \hat{\alpha}\} \,|\, \gamma_1^{\delta_n}] \\ &= \mathbb{1}\{\gamma_1^{\delta_n}(T^{\delta_n}) = x_2^{\delta_n}\}\, \hat{\mathbb{P}}^{\delta_n}[\widehat{\vartheta}^{\delta_n} = \hat{\alpha}] \\ &\overset{n \to \infty}{\longrightarrow} \mathbb{1}\{\gamma(T) = x_2\} \frac{\mathcal{Z}_{\hat{\alpha}}(H_{\gamma_1}; x_3, \ldots, x_{2N})}{\sum_{\beta \in \mathrm{LP}_{N-1}} \mathcal{Z}_\beta(H_{\gamma_1}; x_3, \ldots, x_{2N})}, \end{aligned} \tag{6.6}$$

where $\hat{\mathbb{P}}^{\delta_n}$ is the law of the Ising interfaces on the random polygon $(H^{\delta_n}; x_3^{\delta_n}, \ldots, x_{2N}^{\delta_n})$, measurable with respect to $\gamma_1^{\delta_n}$, which form a random connectivity pattern $\widehat{\vartheta}^{\delta_n} \in \mathrm{LP}_{N-1}$ (and where, by the Skorohod representation theorem, we couple all of the random variables on the same probability space so that the convergence takes place almost surely). Thus, we conclude (using the bounded convergence theorem) that

$$\begin{aligned} \lim_{n\to\infty} \mathbb{P}^{\delta_n}[\vartheta^{\delta_n} = \alpha] &= \lim_{n\to\infty} \mathbb{E}^{\delta_n}\big[\mathbb{1}\{\gamma_1^{\delta_n}(T^{\delta_n}) = x_2^{\delta_n}\}\, \mathbb{E}^{\delta_n}[\mathbb{1}\{\vartheta^{\delta_n} = \alpha\} \,|\, \gamma_1^{\delta_n}]\big] && \text{[by tower property]} \\ &= \mathbb{E}\bigg[\mathbb{1}\{\gamma_1(T) = x_2\} \frac{\mathcal{Z}_{\hat{\alpha}}(H_{\gamma_1}; x_3, \ldots, x_{2N})}{\sum_{\beta \in \mathrm{LP}_{N-1}} \mathcal{Z}_\beta(H_{\gamma_1}; x_3, \ldots, x_{2N})}\bigg] && \text{[by (6.6)]} \\ &= \frac{\mathcal{Z}_\alpha(x_1, \ldots, x_{2N})}{\sum_{\beta \in \mathrm{LP}_N} \mathcal{Z}_\beta(x_1, \ldots, x_{2N})}. && \text{[by (6.5)]} \end{aligned}$$

This completes the induction step and finishes the proof. □

7 Large deviation principle (LDP) for SLE(κ) as $\kappa \to 0+$

When κ is small, the SLE(κ) curves tend to be relatively straight, but still fractal. In the limit $\kappa \to 0$, the SLE(κ) probability measure on the curve space $\mathcal{X}$ concentrates on an atomic measure supported on the unique hyperbolic geodesic connecting the two endpoints of the curve. To make this precise, let us consider the chordal case.

7.1 Large deviation principle

Intuitively, for a given reference curve γ in $\overline{D}$ connecting two boundary points $x, y \in \partial D$, we expect a limiting behavior (large deviation principle,"LDP") of type

$$\text{“}\, \mathbb{P}[\text{SLE}(\kappa) \text{ in } (D; x, y) \text{ stays close to } \gamma] \quad \overset{\kappa \to 0+}{\approx} \quad \exp\Big(-\frac{I_{D;x,y}(\gamma)}{\kappa} \Big) \,\text{”},$$

where $I_{D;x,y}$ is a conformally invariant quantity called *Loewner energy* of the curve γ:

$$I_{D;x,y}(\gamma) := I_{\mathbb{H};0,\infty}(\varphi(\gamma)) := \frac{1}{2}\int_0^\infty |W'(t)|^2\,\mathrm{d}t,$$

where W is the Loewner driving function of $\varphi(\gamma)$, that is the image of γ under any conformal map $\varphi : D \to \mathbb{H}$ sending x and y respectively to 0 and ∞. Note that the right-hand side is the Dirichlet energy of W. The Loewner energy of a chord is not always finite.

As our reference domain, let us choose $D = \mathbb{D}$. For each link pattern $\alpha \in \mathrm{LP}_N$, we endow the curve space $\mathcal{X}_\alpha = \mathcal{X}_\alpha(\mathbb{D}; x_1, \dots, x_{2N}) \subset \prod_j \mathcal{X}^0(\mathbb{D}; x_{a_j}, x_{b_j})$ with the product topology induced by the Hausdorff metric. We will state the LDP for an arbitrary number of chordal curves. For this, we need to also define the multichord energy:

$$I^\alpha_{\mathbb{D}}(\overline{\gamma}) := \Big(\sum_{j=1}^N I_{\mathbb{D};x_{a_j},x_{b_j}}(\gamma_j) + 12\, m_{\mathbb{D}}(\overline{\gamma})\Big) - \inf_{\overline{\gamma}\in\mathcal{X}_\alpha}\Big(\sum_{j=1}^N I_{\mathbb{D};x_{a_j},x_{b_j}}(\gamma_j) + 12\, m_{\mathbb{D}}(\overline{\gamma})\Big) \in [0,+\infty].$$

Theorem 7.1. *The family of laws $(\mathbb{P}^\kappa_\alpha)_{\kappa>0}$ of the multichordal SLE(κ) curves $\overline{\gamma}^\kappa$ satisfies LDP in $\mathcal{X}_\alpha$ with good rate function $I^\alpha_{\mathbb{D}}$, that is, for any closed subset F and open subset O of $\mathcal{X}_\alpha$, we have*

$$\begin{aligned}\limsup_{\kappa\to0+}\kappa\log\mathbb{P}^\kappa_\alpha[\overline{\gamma}^\kappa \in F] &\le -\inf_{\overline{\gamma}\in F} I^\alpha_{\mathbb{D}}(\overline{\gamma}),\\ \liminf_{\kappa\to0+}\kappa\log\mathbb{P}^\kappa_\alpha[\overline{\gamma}^\kappa \in O] &\ge -\inf_{\overline{\gamma}\in O} I^\alpha_{\mathbb{D}}(\overline{\gamma}),\end{aligned}$$

$I^\alpha_{\mathbb{D}}$ is lower semicontinuous, and its level set $(I^\alpha_{\mathbb{D}})^{-1}[0,c]$ is compact for any $c \ge 0$.

Proof idea. See [PW24, Theorem 1.5]. The strategy is as follows:

▷ The case of $N = 1$ is proven using Schilder's theorem for Brownian motion.

▷ To get the case $N \ge 2$ and to understand the meaning of the various terms in the Loewner energy $I^\alpha_{\mathbb{D}}$, recall that $\mathbb{Q}^\kappa_\alpha$ denotes the product measure of N independent chordal SLE(κ) curves associated to the link pattern α, and the multichordal N-SLE(κ) probability measure $\mathbb{P}^\kappa_\alpha$ on $\mathcal{X}_\alpha$ can be obtained by weighting $\mathbb{Q}^\kappa_\alpha$ with the Radon-Nikodym derivative (6.1). Using the case of $N = 1$, the probability measures $(\mathbb{Q}^\kappa_\alpha)_{\kappa>0}$ of independent SLEs satisfy a LDP with rate function just the sum of the independent rate functions,

$$I^\alpha_0(\overline{\gamma}) := \sum_{j=1}^N I_{\mathbb{D};x_{a_j},x_{b_j}}(\gamma_j).$$

From this and the Radon-Nikodym derivative (6.1), we can get to the asserted result by using a tool from large deviations theory (see, e.g., [DZ10, Lemmas 4.3.4 and 4.3.6])):

Lemma 7.2 (Varadhan's lemma)**.** *Suppose that the probability measures $(\mathbb{Q}^\kappa)_{\kappa>0}$ satisfy a LDP with good rate function I^α_0. Let $\Phi : \prod_j \mathcal{X}^0(\mathbb{D}; x_{a_j}, x_{b_j}) \to \mathbb{R}$ be a function bounded from above. Then, the following hold.*

1. *If Φ is upper semicontinuous, then for any closed subset F of $\prod_j \mathcal{X}^0(\mathbb{D}; x_{a_j}, x_{b_j})$,*

$$\limsup_{\kappa\to0}\kappa\log\mathbb{E}^\kappa\Big[\exp\Big(\frac{1}{\kappa}\Phi(\overline{\gamma}^\kappa)\Big)\mathbb{1}\{\overline{\gamma}^\kappa \in F\}\Big] \le -\inf_{\overline{\gamma}\in F}\,(I^\alpha_0(\overline{\gamma}) - \Phi(\overline{\gamma})).$$

2. *If Φ is lower semicontinuous, then for any open subset O of $\prod_j \mathcal{X}^0(\mathbb{D}; x_{a_j}, x_{b_j})$,*

$$\liminf_{\kappa\to0}\kappa\log\mathbb{E}^\kappa\Big[\exp\Big(\frac{1}{\kappa}\Phi(\overline{\gamma}^\kappa)\Big)\mathbb{1}\{\overline{\gamma}^\kappa \in O\}\Big] \ge -\inf_{\overline{\gamma}\in O}\,(I^\alpha_0(\overline{\gamma}) - \Phi(\overline{\gamma})).$$

Using the case of one curve, Varadhan's lemma, and teh construction of multichordal SLE(κ), it is straightforward to derive the LDP for multichordal case, see [PW24, Theorem 5.11] for details. One mainly has to derive some estimates to get rid of the κ-dependence in the function $\Phi_\kappa(\overline{\gamma}))$ in Remark 6.6.

□

In the limit $\kappa \to 0+$ the multichordal SLE(κ) concentrates on minimizers of the energy. In fact, for each connectivity pattern α, there is a unique minimizer (as we will see below).

Corollary 7.3. *As $\kappa \to 0+$, the multichordal SLE(κ) in $(\mathbb{D}; x_1, \dots, x_{2N})$ associated to $\alpha \in \mathrm{LP}_N$ converges in probability to the unique minimizer of the Loewner energy $I^\alpha_\mathbb{D}$ in $\mathcal{X}_\alpha$.*

Proof idea. Suppose $\overline{\eta}$ is the unique minimizer of $I^\alpha_\mathbb{D}$ (due to Theorem 7.4 below). Let $B_\varepsilon(\overline{\eta}) \subset \mathcal{X}_\alpha$ be a Hausdorff-open ball of radius $\varepsilon > 0$ around $\overline{\eta}$. Then, we have

$$\limsup_{\kappa \to 0+} \kappa \log \mathbb{P}^\kappa[\overline{\gamma}^\kappa \in \mathcal{X}_\alpha \smallsetminus B_\varepsilon(\overline{\eta})] \le - \inf_{\overline{\gamma} \in \mathcal{X}_\alpha \smallsetminus B_\varepsilon(\overline{\eta})} I^\alpha_\mathbb{D}(\overline{\gamma}) < 0.$$

This shows the convergence in probability. □

7.2 Finding minimizers of Loewner energy

The unique minimizer $\overline{\eta} = (\eta_1, \dots, \eta_N)$ will have the following property: each η_j is the *hyperbolic geodesic*[10], that is, SLE(0), between the points x_{a_j}, x_{b_j} in the connected component of $\mathbb{D} \smallsetminus \bigcup_{i \ne j} \eta_i$ containing η_j. We call a multichord $\overline{\eta}$ with this property a *geodesic multichord*.

Theorem 7.4. *There exists a unique geodesic multichord $\overline{\eta}$ in $\mathcal{X}_\alpha$ for each α. This multichord $\overline{\eta}$ is the unique minimizer of $I^\alpha_\mathbb{D}$.*

Proof idea. See [PW24, Theorem 1.1 and Corollary 1.6]. The strategy is as follows:

- ▷ From the proof of the LDP (Theorem 7.1), we know that the Loewner energy $I^\alpha_\mathbb{D}$ is lower semicontinuous, which implies that there exist minimizers for it.
- ▷ One can show that any minimizer of $I^\alpha_\mathbb{D}$ is a geodesic multichord. This follows by induction: when $N = 1$, there is just one hyperbolic geodesic, SLE(0), and properties of I^α_D with respect to changing N to $N + 1$ can be used to check that the geodesic multichord property holds for all N. See [PW24, Corollary 3.9].
- ▷ Lastly, we have to show that there are no other geodesic multichords. This is a consequence of the following algebraic result (that holds on $\mathbb{H}$) and conformal invariance (to get to $\mathbb{D}$): each geodesic multichord gives rise to a unique rational function with a given set of $2N$ critical points. See [PW24, Theorem 1.2].
- ▷ To conclude, it remains to classify these rational functions. From algebraic geometry [Gol91] we know that there are *at most* $C_N = \frac{1}{N+1}\binom{2N}{N}$ of them (up to post-composition by a Möbius transformation of the Riemann sphere). Since C_N is also the number of N-link patterns, this yields the uniqueness. □

Theorem 7.5. *Let $\overline{\eta}$ be a geodesic multichord in $\mathcal{X}_\alpha(\mathbb{H}; x_1, \dots, x_{2N})$. The union of $\overline{\eta}$, its complex conjugate, and the real line is the real locus of a real rational function of degree $N+1$ with critical points $\{x_1, \dots, x_{2N}\}$.*

Proof idea. This result is constructive, using Riemann mapping theorem and Schwarz reflection across the curves η_j to obtain the sought rational function. See [PW24, Theorem 1.2]. □

[10] A *hyperbolic geodesic* in $(D; x, y)$ is the image of $[-1, 1]$ by a conformal map $\varphi : \mathbb{D} \to D$ such that $\varphi(1) = x$ and $\varphi(-1) = y$.

As a by-product, we obtain an analytic proof of the *Shapiro conjecture* in real enumerative geometry (first proved by Eremenko & Gabrielov [EG02]): if all critical points of a rational function are real, then the function is real up to post-composition by a Möbius transformation.

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